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RESEARCH ON THE MOTION OF MINOR PLANETS

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RESEARCH ON THE MOTION OF MINOR PLANETS *[/1]

Part I

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Semiconvergent trigonometric series in powers of a small parameter, based on Poincaré's methods, are applied to a qualitative theory on the motion of minor planets and on the origin of gaps in the interplanetary asteroid belt. The asteroids are grouped into characteristic ordinary, singular, regular, and critical planets whose mean absolute motion is compared with that of Jupiter. A formal theory is developed for critical planets of the $(p + q) : p$ type, and the absence of gaps in the asteroid belt at $q \geq 5$ is explained by the derived fact that the motion of "critical" minor planets is stable above this value.

Author

In his authoritative work "New Methods of Celestial Mechanics" (Les Méthodes Nouvelles de la Mécanique Céleste), Poincaré discussed the principles of the formal integration of certain classes of differential equations that are frequently encountered in dynamics. Poincaré's basic concept is the use of semiconvergent trigonometric series, expanded in powers of a small parameter. Because of the canonical form of the equations, always retained by Poincaré, his methods are extremely elegant. In such complex problems as celestial mechanics, elegance and symmetry of the formulas are naturally of major importance.

In Vol.I of his "Lectures on Celestial Mechanics" (Leçons de Mécanique Céleste), Poincaré himself applied these principles to a qualitative study of the motion of the major planets.

The modern theories of the moon, developed in a detailed manner by Delaunay and by M.E.W.Brown, are based on similar principles.

In the research, whose first part is given here, I intend to apply these same principles (with several necessary modifications) to a qualitative theory of minor planets.

Specifically, I intend to investigate cases in which the problem of minor planets can be formally solved by the above-mentioned semiconvergent series. However, I will also specify so-called exceptional cases where entirely new methods must be invented. It will be demonstrated that a study of these excep- /2

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tional cases is intimately connected with the question as to the origin of the gaps in the interplanetary asteroid belt. Incidentally, this is a question, important not only from the analytical viewpoint but also from the cosmogonic viewpoint, which is still far from solution.

At the top of our entire research complex, in Section 1 of this Part I, we will discuss a general method for reducing, as far as possible, the degree of freedom of certain canonical systems of differential equations. In Section 2, we will give differential equations, in a convenient form, on the motion of minor planets.

In the following Sections of this Part, we will assume that the mean motion of a minor planet is not in an approximately commensurable and simple ratio to that of Jupiter. These minor planets have become known as "ordinary" planets ("gewöhnliche" planets according to the terminology of Brendel). First, in Section 3, we will apply the method developed in Section 1 to the theory of these ordinary planets, by eliminating the two mean longitudes. This will yield differential equations of secular inequalities. The entire difficulty of the problem is thus reduced to the integration of a canonical system with two degrees of freedom. In Sections 4, 5, and 6 we will develop secular inequalities in series arranged in accordance with the order of magnitude of the terms, taking into consideration the eccentricities, the inclination, and the square root of the mass of Jupiter, as quantities of the first order of magnitude. The various terms of these series are trigonometric functions of the two arguments w' and w'', which are linear with respect to time. The velocities of the mentioned arguments are of the order of the mass μ of Jupiter, i.e., of the second order of magnitude. However, the velocity of the argument $w' + w''$ is of the fourth order of magnitude (or, occasionally, smaller). All difficulties of the problem result from this fact. It follows from this that the terms composing the expansion of secular inequalities are rational functions with respect to the mass of Jupiter, to the eccentricity of its orbit, and to the moduli of the eccentricity and the inclination of the orbit of the minor planet. The denominators of /3 these rational functions are powers of a certain quantity δ which is homogeneous and linear with respect to the mass of Jupiter, to the square of the eccentricity of its orbit, and to the squares of the moduli of eccentricity and inclination of the orbit of the minor planet. In addition, $\mu\delta$ more or less represents the velocity of the argument $w' + w''$ mentioned above.

Because of the divisor δ which apparently has escaped the attention of scientists, two categories of ordinary minor planets must be differentiated: regular planets for which δ truly is of the second order of magnitude (comparable with μ) and singular planets for which δ is of the third order of magnitude (comparable with $\mu^{3/2}$) or smaller.

The developments of the secular inequalities, formed in Sections 4, 5, and 6, lead to a complete solution of the problem for the case of regular planets. Conversely, for singular planets, the mentioned series can no longer be used.

The theory of singular planets will be treated in Part II of this research.

Later, we will also study the case in which the mean motion of the two planets are more or less at a simple and commensurable ratio. This will finally lead to the problem as to the origin of the gaps in the distribution of the asteroids.

Section 1.

In celestial mechanics, frequently differential equations of the following type are encountered:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \qquad (i = 1, 2, \dots r), \\ \frac{d\xi_k}{dt} &= \frac{dF}{d\eta_k}, \quad \frac{d\eta_k}{dt} = -\frac{dF}{d\xi_k} \qquad (k = 1, 2, \dots s). \end{aligned} \tag{1}$$

The characteristic function F does not depend explicitly on the time t. This function can be expanded in powers of a small parameter μ, so that

$$F(x_i, y_i; \xi_k, \eta_k) = F_0 + \mu F_1 + \mu^2 F_2 + \cdots.$$

The functions F_1, F_2, ... are periodic with the period 2π with respect to the variables y_1, ... y_r and can be expanded in powers of ξ_k and η_k ($k = 1, 2, \dots s$) and do not change on simultaneously changing the signs of the variables y_1, ... y_r ; η_1, ... η_s. Thus, by posing 4

$$\xi_k = \varrho_k \cos \omega_k, \qquad \eta_k = \varrho_k \sin \omega_k \quad (k = 1, 2, \dots s) \tag{2}$$

expansions of the following form will be obtained:

$$F_\nu = \Sigma C \varrho_1^{m_1} \dots \varrho_s^{m_s} \cos(\Sigma p_i y_i + \Sigma q_k \omega_k), \tag{3}$$

where the p_i, q_k, m_k are integers so that $m_k - |q_k|$ is even and nonnegative. The coefficients C depend only on the variables x_1, ... x_r. The first term F_0 has the particularly simple form

$$F_0 = h(x_1, \dots x_r) - \frac{1}{2} \Sigma \nu_k \varrho_k^2, \tag{4}$$

where h is any function of x_1, ... x_r while ν_1, ... ν_s are certain constant coefficients. Thus, F_0 does not depend on the angular variables y_1, ... y_r ; ω_1, ... ω_s.

Posing $\mu = 0$ in eqs.(1), the integration becomes immediate. In this case, x_i, ρ_k are arbitrary constants. In addition, we have $y_i = n_i t + c_i$, $\omega_k = \nu_k t + \gamma_k$ where, again, c_i and γ_k are arbitrary constants. The quantities n_i are given by the formula

$$n_i = -\frac{dF_0}{dx_i} = -\frac{dh}{dx_i}. \tag{5}$$

If the quantity μ is not zero but sufficiently small, it becomes possible to formally satisfy eqs.(1) by means of certain semiconvergent series of a purely trigonometric form. These series were specifically studied by Poincaré. However, the series in question are not valid for all the values of arbitrary constants that enter these series. The difficulty depends on the introduction of small divisors of the form

$$\Sigma p_i n_i + \Sigma q_k \nu_k, \tag{5'}$$

where p_i and q_k are integers. Thus, a complete integration of eqs.(1), from the formal viewpoint, has never been possible. /5

If the integration constants are so selected that no small divisors of the above-indicated form exist (for not too large values of the numbers p_i and q_k), then the integration can be performed by the Lindstedt method.

If only a single small divisor is present, the Bohlin method will be successful.

However, if the number of small divisors is greater than unity, a general method for performing the formal integration is still required. Nevertheless, by means of convenient transformations it becomes occasionally possible to reduce the problem to the Lindstedt case or to the Bohlin case. The theory of minor planets furnishes examples for this.

In the research scheduled by us, it is of importance to reduce, as much as possible, the degree of freedom of the canonical system (1). For this purpose, let us start from the equation of partial derivatives

$$F\left(\frac{dS}{dy_i}, y_i; \frac{dS}{d\eta_k}, \eta_k\right) = F^*\left(x_i, \frac{dS}{dx_i}; \xi_k, \frac{dS}{d\xi_k}\right), \tag{6}$$

where $F(x_i, y_i; \xi_k, \eta_k)$ is the characteristic function of the system (1) while $F^*(x_i, y_i; \xi_k, \eta_k)$ is a new conveniently selected function.

Let

$$S(x_i, y_i; \xi_k, \eta_k)$$

be a particular solution of eq.(6). In eqs.(1), we will substitute the variables $x_i, y_i; \xi_k, \eta_k$ by the new variables $x_i^*, y_i^*; \xi_k^*, \eta_k^*$, defined by the following equations:

$$\begin{aligned} x_i &= \frac{dS(x_i^*, y_i; \xi_k^*, \eta_k)}{dy_i}, & y_i^* &= \frac{dS(x_i^*, y_i; \xi_k^*, \eta_k)}{dx_i^*}, \\ \xi_k &= \frac{dS(x_i^*, y_i; \xi_k^*, \eta_k)}{d\eta_k}, & \eta_k^* &= \frac{dS(x_i^*, y_i; \xi_k^*, \eta_k)}{d\xi_k^*}. \end{aligned} \tag{7}$$

We will then have /6

$$F(x_i, y_i; \xi_k, \eta_k) = F^*(x_i^*, y_i^*; \xi_k^*, \eta_k^*)$$

as well as the new canonical system

$$\frac{dx_i^*}{dt} = \frac{dF^*}{dy_i^*}, \quad \frac{dy_i^*}{dt} = -\frac{dF^*}{dx_i^*} \quad (i = 1, 2, \ldots r)$$
$$\frac{d\xi_k^*}{dt} = \frac{dF^*}{d\eta_k^*}, \quad \frac{d\eta_k^*}{dt} = -\frac{dF^*}{d\xi_k^*} \quad (k = 1, 2, \ldots s) \tag{8}$$

It then is a question of selecting the function F^* in a convenient manner so that no small divisors are present in the function S and so that the degree of freedom of the new system (8) can be diminished.

Let us expand S and F^* in powers of μ by posing

$$\begin{aligned} S &= S_0 + \mu S_1 + \mu^2 S_2 + \cdots, \\ F^* &= F_0^* + \mu F_1^* + \mu^2 F_2^* + \cdots, \end{aligned} \tag{9}$$

and, in the expansions of the two members of eq.(6), let us equate the coefficients of μ^0, of μ, of μ^2, etc.

Setting

$$F_0^* \equiv F_0 \equiv h(x_i) - \frac{1}{2} \Sigma \nu_k (\xi_k^2 + \eta_k^2),$$
$$S_0 = \Sigma x_i y_i + \Sigma \xi_k \eta_k,$$

eq.(6) will be satisfied for $\mu = 0$.

Let us next equate the coefficients of μ in the two members of eq.(6). Making use of the notation (5), this will yield

$$-\Sigma n_i \frac{dS_1}{dy_i} - \Sigma \nu_k \xi_k \frac{dS_1}{d\eta_k} + F_1\,(x_i, y_i; \xi_k, \eta_k)$$
$$= -\Sigma \nu_k \eta_k \frac{dS_1}{d\xi_k} + F_1^*(x_i, y_i; \xi_k, \eta_k).$$

Then, by means of the formulas (2), let us introduce the variables ρ_k, ω_k instead of the variables ξ_k, η_k. This will yield the equation 17

$$\Sigma n_i \frac{dS_1}{dy_i} + \Sigma \nu_k \frac{dS_1}{d\omega_k} = F_1 - F_1^*. \tag{10}$$

In view of eq.(3), we will have an expansion of the following form for F_1:

$$F_1 = \Sigma C \cos (\Sigma p_i y_i + \Sigma q_k \omega_k).$$

After this, we can satisfy eq.(10) by selecting for S_1 and for F_1^* the following expressions:

$$S_1 = \Sigma' \frac{C \sin (\Sigma p_i y_i + \Sigma q_k \omega_k)}{\Sigma p_i n_i + \Sigma q_k \nu_k},$$

$$F_1^* = \Sigma'' C \cos(\Sigma p_i y_i + \Sigma q_k \omega_k).$$

In Σ' we have excluded all terms whose divisors (5') would be small; these terms with small divisors are combined in the sum Σ''.

A given divisor will be considered small if it is of the order $\sqrt{\mu}$ or less. It should be noted here that the divisors are approximately constant. In fact, it will be found that their variations are of the order of μ.

After having selected the functions S_1 and F_1^* in this manner, we will compare the coefficients of μ^2 in the series expansions of the two members of eq.(6). This will yield the equation

$$\Sigma n_i \frac{dS_2}{dy_i} + \Sigma \nu_k \frac{dS_2}{d\omega_k} = \bar{F}_2 - F_2^*, \tag{11}$$

by posing, for abbreviation,

$$\begin{aligned} \bar{F}_2 = F_2 &+ \sum_{i=1}^{r} \frac{dF_1}{dx_i}\frac{dS_1}{dy_i} + \sum_{k=1}^{s} \frac{dF_1}{d\xi_k}\frac{dS_1}{d\eta_k} \\ &- \sum_{i=1}^{r} \frac{dF_1^*}{dy_i}\frac{dS_1}{dx_i} - \sum_{k=1}^{s} \frac{dF_1^*}{d\eta_k}\frac{dS_1}{d\xi_k} \\ &+ \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} \frac{d^2h}{dx_i dx_j}\frac{dS_1}{dy_i}\frac{dS_1}{dy_j} + \frac{1}{2}\sum_{k=1}^{s} \nu_k \left\{\left(\frac{dS_1}{d\xi_k}\right)^2 - \left(\frac{dS_1}{d\eta_k}\right)^2\right\}. \end{aligned} \tag{12}$$

For F_2^*, we will select the ensemble of the terms of the trigonometric expansion for $\bar{F}_2$, whose divisors are assumed as small. After having selected F_2^* in this manner, eq.(11) can be satisfied by a function S_2 whose expansion contains no terms enlarged by the integration. /8

Obviously, we can continue in this manner and thus completely determine the various terms of the expansions (9).

Let us investigate the new canonical system (8) in more detail. In analogy to the formulas (2), let us put

$$\xi_k^* = \rho_k^* \cos \omega_k^*, \quad \eta_k^* = \rho_k^* \sin \omega_k^*.$$

The function F^*, expanded in multiples of the arguments y_i^* and ω_k^* contains only arguments with small divisors. Let

$$\sum_{i=1}^{r} a_i^{(j)} n_i + \sum_{k=1}^{s} \alpha_k^{(j)} \nu_k \qquad (j = 1, 2, \ldots m)$$

be the small divisors that are linearly independent (where $a_i^{(j)}$ and $\alpha_k^{(k)}$ are given integers). If the parameter μ is sufficiently small and if the quantities n_i and ν_k are not extremely small (of the order of $\sqrt{\mu}$ or smaller), we certainly will have $m < r + s$. All the arguments

$$\Sigma p_i y_i^* + \Sigma q_k \omega_k^*$$

which appear in $F^*(x_i^*, y_i^*; \xi_k^*, \eta_k^*)$ are then linearly composed of m arguments

$$\sum_{i=1}^{r} a_i^{(j)} y_i^* + \sum_{k=1}^{s} \alpha_k^{(j)} \omega_k^* \quad (j = 1, 2, \dots m).$$

Similarly, the partial derivatives $\frac{dF^*}{dy_i^*}$ and $\frac{dF^*}{d\omega_k^*}$ are mutually and linearly composed of m.

After this and in view of the fact that the canonical system (8) can be replaced by the system /9

$$\frac{dx_i^*}{dt} = \frac{dF^*}{dy_i^*}, \quad \frac{dy_i^*}{dt} = -\frac{dF^*}{dx_i^*},$$
$$\frac{d\frac{1}{2}\varrho_k^{*2}}{dt} = \frac{dF^*}{d\omega_k^*}, \quad \frac{d\omega_k^*}{dt} = -\frac{dF^*}{d\frac{1}{2}\varrho_k^{*2}}, \tag{13}$$

it is obvious that we will have $r + s - m$ first integrals which are linear and homogeneous expressions with respect to the variables x_i^* and $\frac{1}{2}\rho_k^{*2}$ with integral coefficients. Thus, the degree of freedom of the system (8) or (13) can be lowered to m.

It is often possible, by means of convenient transformations, to place the reduced system in the form (1) and to continue the reduction in this manner until the degree of freedom becomes zero or one. This will then permit formal integration of the system (1).

Section 2.

Here, we will apply the above reduction method to the general problem of minor planets.

Primarily, it is necessary to bring the equations of motion to the form of eq.(1).

Let us consider a minor planet of infinitely small mass, moving under the attraction of the Sun and of Jupiter. Let us assume that the planet Jupiter moves in accordance with Kepler's laws in an ellipse with an eccentricity e'.

Let x, y, z be the rectangular coordinates of the minor planet in a system of coordinates whose origin is located at the center of the Sun and whose z-axis is perpendicular to the orbit of Jupiter. Let r and r_1 be the radius vectors of

the planetoid and of Jupiter. As unit length, let us select the semimajor axis of the orbit of Jupiter and, as unit mass, the sum of the masses of both Sun and Jupiter, and let us fix the unit time in such a manner that the gravitational constant will be equal to 1. Let, finally, μ be the mass of Jupiter and H the angle to the Sun between the radius vectors directed toward the minor planet /10
and toward Jupiter. Then, the equations of motion of the minor planet can be written as follows:

$$\frac{dx}{dt} = -\frac{d\tilde{F}}{dx'}, \quad \frac{dx'}{dt} = \frac{d\tilde{F}}{dx},$$
$$\frac{dy}{dt} = -\frac{d\tilde{F}}{dy'}, \quad \frac{dy'}{dt} = \frac{d\tilde{F}}{dy}, \qquad (1)$$
$$\frac{dz}{dt} = -\frac{d\tilde{F}}{dz'}, \quad \frac{dz'}{dt} = \frac{d\tilde{F}}{dz},$$

where

$$\tilde{F} = \tilde{F}_0 + \mu \tilde{F}_1,$$
$$\tilde{F}_0 = -\frac{1}{2}(x'^2 + y'^2 + z'^2) + \frac{1}{r},$$
$$\tilde{F}_1 = \frac{1}{\sqrt{r_1^2 - 2r_1 r \cos H + r^2}} - \frac{r \cos H}{r_1^2} - \frac{1}{r}.$$

[See, for example, Tisserand, (Bibl.1).]

As variables, we will introduce "canonical elements" defined in the following manner: Imagine a moving point of mass *one*, attracted toward a fixed center by a force of the magnitude $\frac{1}{r^2}$ where r is the distance of the moving point from the fixed center. In this case, the moving point describes a conic section about the fixed center as focus, in accordance with Kepler's laws. Let us assume that the orbit is an ellipse and that a, e, I, ℓ, g, θ are its Keplerian elements such that a is the semimajor axis, e the eccentricity, I the inclination, ℓ the mean anomaly of the moving point, g the distance of the perihelion to the ascending node, and θ the longitude of this node. The coordinates x, y, z of the moving point with respect to the fixed center and its velocity components x', y', z' are then certain well-known functions of the Keplerian elements a, e, I, ℓ, g, θ or else of the canonical elements L, G, Θ, ℓ, g, θ whose three first are given by the formulas

$$L = \sqrt{a}, \quad G = \sqrt{a(1-e^2)}, \quad \Theta = \sqrt{a(1-e^2)} \cos I$$

On substituting, in eqs.(1), the variables x, y, z, x', y', z' by the /11
canonical elements L, G, Θ, ℓ, g, θ, the equations will become

$$\frac{dL}{dt} = \frac{d\tilde{F}}{dl}, \quad \frac{dl}{dt} = -\frac{d\tilde{F}}{dL},$$
$$\frac{dG}{dt} = \frac{d\tilde{F}}{dg}, \quad \frac{dg}{dt} = -\frac{d\tilde{F}}{dG}, \qquad (2)$$

$$\frac{d\Theta}{dt}=\frac{d\tilde{F}}{d\theta}, \quad \frac{d\theta}{dt}=-\frac{d\tilde{F}}{d\Theta}.$$

Since the integral of the vis viva in the Keplerian motion is given, we will have

$$\tilde{F}_0=-\frac{1}{2}(x'^2+y'^2+z'^2)+\frac{1}{r}=\frac{1}{2a}=\frac{1}{2L^2}.$$

In accordance with the above selection of units, the mean motion of Jupiter is equal to 1. We will count the longitudes starting from the perihelion of Jupiter. By counting the time from the passage of Jupiter through its perihelion, the mean longitude (or mean anomaly) of this planet will be t. The perturbing function $\tilde{F}_1$ is periodic, with the period 2π, with respect to the angular variables l, g, θ, t.

We will first substitute the variables

$$\begin{matrix} L, & G, & \Theta, \\ l, & g, & \theta \end{matrix}$$

by

$$x_1=L, \quad \frac{1}{2}\rho_1^2=L-G, \quad \frac{1}{2}\rho_2^2=G-\Theta,$$
$$y_1=l+g+\theta, \quad \omega_1=-g-\theta, \quad \omega_2=-\theta$$

and then by

$$x_1, \quad \xi_1=\rho_1\cos\omega_1, \quad \xi_2=\rho_2\cos\omega_2,$$
$$y_1, \quad \eta_1=\rho_1\sin\omega_1, \quad \eta_2=\rho_2\sin\omega_2.$$

It is well known that these changes in the variables retain the canonical form /12
of eqs.(2).

Finally, to obtain the form of eq.(1) in Section 1, we will introduce the auxiliary variables x_2 and y_2 defined by the formulas

$$y_2=t, \quad \frac{dx_2}{dt}=\frac{d\tilde{F}}{dt}.$$

This will yield the canonical system

$$\left.\begin{aligned} \frac{dx_k}{dt}&=\frac{dF}{dy_k}, & \frac{dy_k}{dt}&=-\frac{dF}{dx_k}, \\ \frac{d\xi_k}{dt}&=\frac{dF}{d\eta_k}, & \frac{d\eta_k}{dt}&=-\frac{dF}{d\xi_k}, \end{aligned}\right\} \quad k=1,2 \qquad (3)$$

where

$$F=\tilde{F}-x_2=F_0+\mu F_1,$$

$$F_0 = \frac{1}{2x_1^2} - x_2,$$

$$F_1 = \frac{1}{\sqrt{r_1^2 - 2r_1 r \cos H + r^2}} - \frac{r \cos H}{r_1^2} - \frac{1}{r}. \tag{4}$$

Ordinarily, the perturbing function F_1 is expanded in a trigonometric series in accordance with multiples of the longitudes

$$y_1 + \omega_2, \quad y_2 + \omega_2, \quad -\omega_1 + \omega_2, \quad \omega_2$$

of the two planets and of their perihelions, starting from the node. This expansion has the form

$$\begin{aligned} F_1 &= \Sigma A e^m e'^{m'} \left(\sin \frac{1}{2} I\right)^{2f} \cos [i_1 (y_1 + \omega_2) + i_2 (y_2 + \omega_2) \\ &\qquad - j_1(-\omega_1 + \omega_2) - j'_2 \omega_2] \\ &= \Sigma A e^m e'^{m'} \left(\sin \frac{1}{2} I\right)^{2f} \cos (i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2), \end{aligned}$$

by setting
$$j'_2 = i_1 + i_2 - j_1 - j_2.$$

The differences $m - |j_1|$, $m' - |j'_2|$, $2f - |j_2|$ are even and ≥ 0 [see Tisserand (Bibl.2)]. The expressions of the coefficients A, as functions of a, had been /13 given by Le Verrier and Newcomb.

However, it is necessary to introduce the canonical elements x_1, ρ_1, and ρ_2 instead of the Keplerian elements a, e, I. Since the formulas that express x_1, ρ_1, and ρ_2 as functions of a, e, I are given, we obtain

$$a = x_1^2, \quad e = \frac{\rho_1}{\sqrt{x_1}} \sqrt{1 - \frac{\rho_1^2}{4x_1}}, \quad \sin^2 \frac{I}{2} = \frac{\rho_2^2}{4x_1 - 2\rho_1^2}.$$

By expanding in powers of ρ_1, ρ_2, and e', we finally obtain the series

$$F_1 = \Sigma F^{l_1, \bar{m}, m_1, m_2}_{i_1, i_2, j_1, j_2} e'^{m} \rho_1^{m_1} \rho_2^{m_2} \cos (i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2). \tag{5}$$

Here, m_2 and j_2 are even numbers; we still have

$$\begin{aligned} m_1 &= |j_1| + 2k_1, \\ m_2 &= |j_2| + 2k_2, \\ \bar{m} &= |i_1 + i_2 - j_1 - j_2| + 2\bar{k}, \\ \bar{m} + m_1 + m_2 &= |i_1 + i_2| + 2k, \end{aligned} \tag{6}$$

where k_1, k_2, $\bar{k}$, and k are positive whole numbers or zero.

The coefficients $F^{1,m,m_1,m_2}_{i_1,i_2,j_1,j_2}$ depend on x_1 but are independent of x_2 (x_2 appears only in F_0). Obviously, we can assume that

$$F^{1,m,m_1,m_2}_{-i_1,-i_2,-j_1,-j_2} = F^{1,m,m_1,m_2}_{i_1,i_2,j_1,j_2}.$$

Section 3.

We will now apply the method given in Section 1 to the system (3) of the preceding Section.

In the actual case, we have r = s = 2 and

$$\nu_1 = \nu_2 = 0,$$

$$h(x_1, x_2) = \frac{1}{2x_1^2} - x_2,$$

$$n_1 = x_1^{-3}, \quad n_2 = 1.$$

Assuming that $\mu = 0$, the quantity n_1 is nothing else but the mean motion /14 of the minor planet whose motion will thus be of the Keplerian type.

In Part I of this report, we will assume that n_1 is not very close to a simple rational number. Thus, we will first discuss the theory of "ordinary" planets.

In this case, the arguments with small divisors are those that are independent of y_1 and y_2.

Let us assume that, in accordance with the rules laid down in Section 1, the functions $S(x_k, y_k; \xi_k, \eta_k)$ and $F^*(x_k; \xi_k, \eta_k)$ as well as the canonical transformation that corresponds to the function $S(x_k^*, y_k; \xi_k^*, \eta_k)$ are formed. The function $F^*(x_k^*; \xi_k^*, \eta_k^*)$ is independent of the arguments y_1^* and y_2^*. The new canonical system, formed with the characteristic function $F^*(x_k^*; \xi_k^*, \eta_k^*)$, thus will have the two following integrals:

$$x_1^* = \text{const.}, \quad x_2^* = \text{const.}$$

The variables ξ_k^*, η_k^* satisfy the equations

$$\frac{d\xi_k^*}{dt} = \frac{dF^*}{d\eta_k^*}, \quad \frac{d\eta_k^*}{dt} = -\frac{dF^*}{d\xi_k^*} \qquad (k = 1, 2). \tag{1}$$

After integrating these equations, we obtain the arguments y_1^* and y_2^* after quadratures by means of the formulas

$$\frac{dy_k^*}{dt} = -\frac{dF^*}{dx_k^*} \qquad (k = 1, 2). \tag{2}$$

Equations (1) and (2) are known as equations of secular variations.

Before proceeding to their integration, it is preferable to make a more detailed study of the functions S and F^* as well as of the canonical transformation which corresponds to $S(x_k^*, y_k; \xi_k^*, \eta_k)$. Let us put, as in Section 1,

$$F^* = F_0^* + \mu F_1^* + \mu^2 F_2^* + \cdots,$$
$$S \;= S_0 \;+ \mu S_1 \;+ \mu^2 S_2 \;+ \cdots.$$

First, we have /15

$$F_0^* = F_0 = \frac{1}{2x_1^2} - x_2, \tag{3}$$

$$S_0 = \sum_{k=1}^{2} (x_k y_k + \xi_k \eta_k).$$

Then, we must put

$$F_1^* = \Sigma F^{1,\bar m, m_1, m_2}_{0,0,j_1,j_2} e'^{\bar m} \varrho_1^{m_1} \varrho_2^{m_2} \cos (j_1 \omega_1 + j_2 \omega_2), \tag{4}$$

$$S_1 = \Sigma' \frac{F^{1,\bar m, m_1, m_2}_{i_1,i_2,j_1,j_2}}{i_1 n_1 + i_2 n_2} e'^{\bar m} \varrho_1^{m_1} \varrho_2^{m_2} \sin (i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2), \tag{5}$$

while excluding in Σ' all the terms where $i_1 = i_2 = 0$. We recall that the relations (6) of Section 2 are still valid.

In continuing, it is first necessary to form the function $\bar F_2$ in accordance with the general formula (12) of Section 1. We thus have

$$\bar F_2 = \frac{3}{2} x_1^{-4} \left(\frac{dS_1}{dy_1}\right)^2 + \frac{dF_1}{dx_1}\frac{dS_1}{dy_1} + \sum_{k=1}^{2} \left(\frac{dF_1}{d\xi_k}\frac{dS_1}{d\eta_k} - \frac{dF_1^*}{d\eta_k}\frac{dS_1}{d\xi_k}\right). \tag{6}$$

Since the form of the functions F_1, F_1^*, and S_1 as well as the conditions (6) of Section 2 are given, it is obvious that $\bar F_2$ will have the same form as F_1 [eq.(5) in Section 2], with the only exception that, in the conditions (6) of Section 2, we must replace $\bar m$ by $\bar m + 2$. When this is done, F_2^* will be the sum of the terms of $\bar F_2$ where $i_1 = i_2 = 0$; finally, S_2 is obtained, after integration, from the formula

$$\Sigma n_k \frac{dS_2}{dy_k} = \bar F_2 - F_2^*,$$

analogous to eq.(11) of Section 1.

We can continue in this manner and thus successively form the functions $\bar F_i$, F_i^*, and S_i. This will yield /16

$$\bar F_i = \Sigma F^{i,\bar m, m_1, m_2}_{i_1,i_2,j_1,j_2} e'^{\bar m} \varrho_1^{m_1} \varrho_2^{m_2} \cos (i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2), \tag{7}$$

$$F_i^* = \sum F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \cos (j_1 \omega_1 + j_2 \omega_2), \tag{8}$$

$$S_i = -\sum' \frac{F_{i_1,i_2,j_1,j_2}^{i,\bar{m},m_1,m_2}}{i_1 n_1 + i_2 n_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \sin (i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2), \tag{9}$$

with the conditions

$$\begin{aligned} m_1 &= |j_1| + 2k_1, \\ m_2 &= |j_2| + 2k_2, \qquad m_2 \text{ and } j_2 \text{, even} \\ \bar{m} + 2i - 2 &= |i_1 + i_2 - j_1 - j_2| + 2\bar{k} \end{aligned} \tag{10}$$

for $\bar{F}_1$ and S_1, and with the conditions

$$\begin{aligned} m_1 &= |j_1| + 2k_1, \\ m_2 &= |j_2| + 2k_2, \qquad m_2 \text{ and } j_2 \text{, even} \\ \bar{m} + 2i - 2 &= |j_1 + j_2| + 2\bar{k} \end{aligned} \tag{11}$$

for F_1^*.

In eqs.(10) and (11), the quantities k_1, k_2, and $\bar{k}$ are nonnegative integers.

Obviously, the auxiliary variable x_2 does not enter the expansions (7), (8), and (9).

Let us now pass to the canonical transformation which, in accordance with the general formulas (7) of Section 1, corresponds to the function $S(x_k^*, y_k; \xi_k^*, \eta_k)$. We can write it in the form of

$$\begin{aligned} x_1 - x_1^* &= \frac{d(S-S_0)}{dy_1}, \quad y_1 - y_1^* = -\frac{d(S-S_0)}{dx_1^*}, \\ \xi_k - \xi_k^* &= \frac{d(S-S_0)}{d\eta_k}, \quad \eta_k - \eta_k^* = -\frac{d(S-S_0)}{d\xi_k^*}. \end{aligned} \tag{12}$$

On the one hand, we have excluded the relation that yields $x_2 - x_2^*$ since we no longer require the auxiliary variable x_2 and, on the other hand, the relation $y_2 - y_2^* = 0$ which shows that

$$y_2^* = y_2 = t. \tag{13}$$

This leaves eqs.(12) to be solved with respect to the variables x_1, y_1; ξ_k, η_k. Obviously, the differences $x_1 - x_1^*$, $y_1 - y_1^*$, $\xi_k - \xi_k^*$, $\eta_k - \eta_k^*$ can be expanded in powers of μ, e', ξ_k^*, η_k^* as well as in multiples of the arguments y_1^* and $y_2^* = t$. Finally, setting $\xi_k^* = \rho_k^* \cos \omega_k^*$, $\eta_k^* = \rho_k^* \sin \omega_k^*$, we obtain expansions of the following form: /17

$$\sum C \mu^i e'^{\bar{m}} \varrho_1^{*m_1} \varrho_2^{*m_2} \begin{matrix}\cos\\ \sin\end{matrix} (i_1 y_1^* + i_2 y_2^* + j_1 \omega_1^* + j_2 \omega_2^*). \tag{14}$$

Here, we have cos for $x_1 - x_1^*$ and $\xi_k - \xi_k^*$; sin for $y_1 - y_1^*$ and $\eta_k - \eta_k^*$.

Moreover, we also have there $m_1 = |j_1| + 2k_1$, $m_2 = |j_2| + 2k_2$ (where k_1 and k_2 are nonnegative integers). Also, m_2 and j_2 are even in the expansions for $x_1 - x_1^*$, $y_1 - y_1^*$, $\xi_1 - \xi_1^*$, and $\eta_1 - \eta_1^*$ while m_2 and j_2 are odd in the series for $\xi_2 - \xi_2^*$ and $\eta_2 - \eta_2^*$. Finally, we have

$$\bar{m} + 2i - 2 = |i_1 + i_2 - j_1 - j_2| + 2\bar{k} \qquad \bar{k} \geq 0 \tag{15}$$

in the expansions for $x_1 - x_1^*$ and $y_1 - y_1^*$, but

$$\bar{m} + 2i - 1 = |i_1 + i_2 - j_1 - j_2| + 2\bar{k} \qquad \bar{k} \geq 0 \tag{15'}$$

in the expansions for $\xi_k - \xi_k^*$ and $\eta_k - \eta_k^*$.

Let us now return to eqs.(1). First, we have a particular solution in which the ξ_k^*, η_k^* are constants. This solution is obtained by canceling the second terms of eqs.(1). In this solution, we have

$$\xi_1^* = \bar{\xi}, \ \eta_1^* = \xi_2^* = \eta_2^* = 0$$

with $\bar{\xi}$ being the root of the equation

$$\frac{d(F^* - F_0^*)}{d\xi_1^*} = 0, \tag{16}$$

where, in the first term, we have put

$$\eta_1^* = \xi_2^* = \eta_2^* = 0.$$

The first term of this equation is an odd function of e' and ξ_1^* since the general function $F^* - F_0^*$ is even with respect to the variables e', ξ_1^*, η_1^*, ξ_2^*, η_2^*. Thus, the quantity $\bar{\xi}:e'$ can be expanded in powers of e'^2 and μ. By only writing the term independent of e'^2 and μ, we obtain [see eq.(4)] /18

$$\bar{\xi} : e' = -\frac{F_{0,0,1,0}^{1,1,1,0}}{F_{0,0,0,0}^{1,0,2,0}} + \cdots.$$

In the theory of minor planets, we can consider the quantities e', e, and $\sin \frac{1}{2} I$ as being comparable, in magnitude, with $\sqrt{\mu}$ (μ being approximately $\approx 10^{-3}$). The same is true for ρ_1, ρ_2, ρ_1^*, and ρ_2^*.

In view of this, for discussing the equations of secular variations (1), it is convenient to put

$$\begin{aligned} &\xi_1^* = \bar{\xi} + \sqrt{\mu}\,\xi' = \sqrt{\mu}\,(\xi_0 + \xi'), \quad \eta_1^* = \sqrt{\mu}\,\eta', \quad e' = \sqrt{\mu}\,e_0, \\ &\xi_2^* = \sqrt{\mu}\,\xi'', \qquad\qquad\qquad\qquad \eta_2^* = \sqrt{\mu}\,\eta'', \quad t_1 = \mu t \end{aligned} \tag{17}$$

and to consider the constants ω_0 and $\xi_0 = \bar{\xi} : \sqrt{\mu}$ as well as the variables ξ', η', ξ'', η'' as quantities comparable, in magnitude, with unity.

The new variables ξ', η', ξ'', η'' satisfy the equations

$$\begin{aligned} \frac{d\xi'}{dt_1} &= \frac{dH}{d\eta'}, & \frac{d\eta'}{dt_1} &= -\frac{dH}{d\xi'}, \\ \frac{d\xi''}{dt_1} &= \frac{dH}{d\eta''}, & \frac{d\eta''}{dt_1} &= -\frac{dH}{d\xi''}, \end{aligned} \tag{18}$$

where

$$H = \frac{F^* - \text{const.}}{\mu^2}. \tag{19}$$

We select the constant such that H is canceled out for $\xi' = \eta' = \xi'' = \eta'' = 0$.

We will investigate the function H in more detail. For this purpose, let us write the expansion of the function $F^* - F_0^*$ in the form of /19

$$F^* - F_0^* = \Sigma F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2} \mu^i e'^{\bar{m}} \varrho_1^{*m_1} \varrho_2^{*m_2} \cos(j_1\omega_1^* + j_2\omega_2^*) \tag{20}$$

with the conditions (11).

We will here introduce new variables, by putting

$$\begin{aligned} \varphi_k^* &= \xi_k^* + \sqrt{-1}\,\eta_k^* = \varrho_k^* e^{\sqrt{-1}\,\omega_k^*}, \\ \psi_k^* &= \xi_k^* - \sqrt{-1}\,\eta_k^* = \varrho_k^* e^{-\sqrt{-1}\,\omega_k^*}. \end{aligned} \tag{21}$$

By setting also $m_k = \alpha_k + \beta_k$, $j_k = \alpha_k - \beta_k$, the expansion (20) can be written as follows:

$$F^* - F_0^* = \Sigma F_{0,0,\alpha_1-\beta_1,\alpha_2-\beta_2}^{i,\bar{m},\alpha_1+\beta_1,\alpha_2+\beta_2} \mu^i e'^{\bar{m}} \varphi_1^{*\alpha_1} \psi_1^{*\beta_1} \varphi_2^{*\alpha_2} \psi_2^{*\beta_2}. \tag{20'}$$

The nonnegative whole numbers α_1, β_1, α_2, β_2 take only the values that satisfy the conditions (11) which here become

$$\begin{aligned} &\alpha_2 + \beta_2 = \text{even}, \\ &\bar{m} + 2i - 2 = |\alpha_1 - \beta_1 + \alpha_2 - \beta_2| + 2\bar{k}. \end{aligned} \tag{22}$$

In view of this, we will put, in analogy with the formulas (17),

$$\begin{aligned} \varphi_1^* &= \sqrt{\mu}(\xi_0 + \varphi'), & \psi_1^* &= \sqrt{\mu}(\xi_0 + \psi'), \\ \varphi_2^* &= \sqrt{\mu}\,\varphi'', & \psi_2^* &= \sqrt{\mu}\,\psi'', \end{aligned} \tag{23}$$

from which, between the variables φ', ψ', φ'', ψ'' and the variables ξ', η', ξ'', η'', the following relations are obtained:

$$\begin{aligned} \varphi' &= \xi' + \sqrt{-1}\,\eta', \quad \psi' = \xi' - \sqrt{-1}\,\eta', \\ \varphi'' &= \xi'' + \sqrt{-1}\,\eta'', \quad \psi'' = \xi'' - \sqrt{-1}\,\eta''. \end{aligned} \tag{24}$$

On introducing, in eq.(20'), the expressions (23) as well as the expression $e' = \sqrt{\mu}\,e_0$, we will obtain in accordance with eq.(19) /20

$$\begin{aligned} H = \Sigma' (\sqrt{\mu})^{2i+\bar{m}+\alpha_1+\beta_1+\alpha_2+\beta_2-4} F^{i,\bar{m},\alpha_1+\beta_1,\alpha_2+\beta_2}_{0,0,\alpha_1-\beta_1,\alpha_2-\beta_2} \\ \cdot e_0^{\bar{m}} (\xi_0 + \varphi')^{\alpha_1} (\xi_0 + \psi')^{\beta_1} \varphi''^{\alpha_2} \psi''^{\beta_2}, \end{aligned} \tag{25}$$

where, in Σ', we must exclude the term constant in and the terms linear in φ', ψ', φ'', and ψ''.

In accordance with our statement on the quantity $\bar{\xi}$ (on p.14) it is obvious that the quantity $(\xi_0 : e_0)^j$ can be expanded in powers of e'^2 and μ. By posing $e'^2 = \mu e_0^2$ and by expanding in powers of μ, we obtain

$$\xi_0^j = e_0^j \sum_{s=0}^{\infty} p_s^{(j)} \mu^s \qquad j = 0, 1, 2, \ldots, \tag{26}$$

denoting by $p_s^{(j)}$ a certain polynomial in e_0^2 of the degree s. Obviously, we then have $p_0^{(0)} \equiv 1$, $p_1^{(0)} \equiv p_2^{(0)} \equiv \ldots \equiv 0$.

Therefore, let us expand, in the expression (25) of H, the quantity $(\xi_0 + \varphi')^{\alpha_1} (\xi_0 + \psi')^{\beta_1}$ in powers of φ', ψ', and ξ_0 and then introduce there the expressions (26) for ξ_0^j. Finally, let us arrange the expression H in powers of μ, φ', ψ', φ'', ψ''. This will yield

$$H = H^{(0)} + \mu H^{(1)} + \mu^2 H^{(2)} + \cdots = \sum_{m=0}^{\infty} \mu^m H^{(m)}, \tag{27}$$

$$H^{(m)} = \sum_{\alpha',\beta',\alpha'',\beta''} H^{m,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''} \varphi'^{\alpha'} \psi'^{\beta'} \varphi''^{\alpha''} \psi''^{\beta''}. \tag{28}$$

The coefficients of this expansion (28) are given by the general formula

$$H^{m,m',m''}_{j',j''} = \sum_{i,\bar{m},\alpha_1,\beta_1,s} \binom{\alpha_1}{\alpha'} \binom{\beta_1}{\beta'} F^{i,\bar{m},m_1,m''}_{0,0,j_1,j''} e_0^{\bar{m}+m_1-m'} p_s^{(m_1-m')}, \tag{29}$$

where we have used, for abbreviation,

$$\begin{aligned} m_1 &= \alpha_1 + \beta_1, \quad m' = \alpha' + \beta', \quad m'' = \alpha'' + \beta'', \\ j_1 &= \alpha_1 - \beta_1, \quad j' = \alpha' - \beta', \quad j'' = \alpha'' - \beta''. \end{aligned}$$

The whole numbers i, $\bar{m}$, α_1, β_1, s in the sum (29) must take values that satisfy the conditions /21

$$\begin{aligned} &i\geq 1,\quad \bar{m}\geq 0,\quad \alpha_1\geq\alpha',\quad \beta_1\geq\beta',\quad s\geq 0,\\ &|\alpha_1-\beta_1+\alpha''-\beta''|\leq \bar{m}+2i-2=2m+2-(\alpha_1+\beta_1+\alpha''+\beta'')-2s. \end{aligned} \tag{30}$$

The nonnegative integers α', β', α'', β'' in the sum (28) must satisfy the relations

$$\begin{aligned} &\alpha'+\beta'+\alpha''+\beta''\geq 2,\quad \alpha''+\beta''\doteq \text{even},\\ &2m+2=|\alpha'-\beta'+\alpha''-\beta''|+\alpha'+\beta'+\alpha''+\beta''+2k\quad (k\geq 0) \end{aligned} \tag{31}$$

where the latter relation is a direct consequence of the conditions (30).

Equation (29) demonstrates that

$$H^{m,m',m''}_{-j',-j''}=H^{m,m',m''}_{j',j''}. \tag{32}$$

To avoid imaginary variables, we will put in eq.(28):

$$\begin{aligned} &\varphi'=\varrho' e^{\sqrt{-1}\,\omega'},\quad \psi'=\varrho' e^{-\sqrt{-1}\,\omega'},\\ &\varphi''=\varrho'' e^{\sqrt{-1}\,\omega''},\quad \psi''=\varrho'' e^{-\sqrt{-1}\,\omega''}, \end{aligned} \tag{33}$$

from which it follows that

$$\begin{aligned} &\xi'=\varrho'\cos\omega',\quad \eta'=\varrho'\sin\omega',\\ &\xi''=\varrho''\cos\omega'',\quad \eta''=\varrho''\sin\omega''. \end{aligned} \tag{34}$$

Thus, the expansion (28) can be written in the form

$$H^{(m)}=\sum_{m',m'',j',j''} H^{m,m',m''}_{j',j''}\varrho'^{m'}\varrho''^{m''}\cos(j'\omega'+j''\omega''). \tag{35}$$

The integrals m', m'', j', j'' take only the values that satisfy the conditions

$$\begin{aligned} &m'+m''\geq 2,\qquad m''=\text{an even number},\\ &m'=|j'|+2k',\quad m''=|j''|+2k'',\\ &2m+2=|j'+j''|+m'+m''+2k\\ &\qquad\quad =|j'+j''|+|j'|+|j''|+2\tilde{k}, \end{aligned} \tag{36}$$

/22

where k', k'', k, and $\tilde{k}$ are any nonnegative integers. This results from the relations (31).

Let us, specifically, investigate the function $H^{(0)}$. For $m = 0$ we will have, because of the conditions (36), either $m' = 2$, $j' = 0$, $m'' = j'' = 0$ or $m' = j' = 0$, $m'' = 2$, $j'' = 0$. Hence,

$$H^{(0)}=H^{0,2,0}_{0,0}\varrho'^2+H^{0,0,2}_{0,0}\varrho''^2.$$

In addition, an application of eq.(29) demonstrates that

$$H_{0,0}^{0,2,0} = F_{0,0,0,0}^{1,0,2,0}, \quad H_{0,0}^{0,0,2} = F_{0,0,0,0}^{1,0,0,2}. \tag{37}$$

We will express these quantities by means of Laplace coefficients. For this purpose, let us investigate the terms of the second degree in the expansion of the function F_1^*. These are (assuming $e' = 0$):

$$F_{0,0,0,0}^{1,0,2,0}\varrho_1^{*2} + F_{0,0,0,0}^{1,0,0,2}\varrho_2^{*2}.$$

The terms of the second degree of the secular part of the perturbative function, according to Tisserand (Bibl.3) are as follows (assuming the eccentricity and the inclination of the perturbing planet as zero):

$$\frac{1}{8a'}\alpha c^{(1)}[e^2 - \mathrm{tg}^2\varphi].$$

By setting there $a' = 1$, $\alpha = a = x_1^{*2}$, $\varphi = I$, $e^2 = \rho_1^{*2}:x_1^*$, $4\sin^2\frac{I}{2} = \rho_2^{*2}:x_1^*$ and by comparing the coefficients of the two expressions of the investigated part of F_1^*, we will find the following formulas: /23

$$F_{0,0,0,0}^{1,0,2,0} = -F_{0,0,0,0}^{1,0,0,2} = \frac{\sqrt{a}}{8}c^{(1)}, \tag{38}$$

where $c^{(1)}$ is the Laplace coefficient calculated with the value $a = x_1^{*2}$.

Thus, on introducing the notations

$$\nu_0' = -\frac{\sqrt{a}}{4}c^{(1)}, \quad \nu_0'' = +\frac{\sqrt{a}}{4}c^{(1)}, \tag{39}$$

we will finally have

$$H^{(0)} = -\frac{\nu_0'}{2}\varrho'^2 - \frac{\nu_0''}{2}\varrho''^2 \tag{40}$$

with the fundamental relation

$$\nu_0' + \nu_0'' = 0. \tag{41}$$

Section 4.

We now have been returned to the system (18) of Section 3, which is a system of the type (1) in Section 1. There are no variables in existence that correspond to x_1, y_1; the variables ξ_k, η_k are denoted here by ξ', η', ξ'', η''. Finally, the quantities ν_0' and ν_0'' of eqs.(39) and (40) in Section 3 correspond to the quantities ν_k of Section 1.

These quantities ν_0' and ν_0'' are not small but they are interrelated by the identical relation (41) of Section 3. Thus, the small divisors are identically zero, and the corresponding arguments are multiples of the argument $2w' + 2w''$.

Again, the method given in Section 1 can be applied here. The new canonical system resulting from this method is readily reduced to one degree of freedom. A complete integration of the system (18) in Section 3, from the formal viewpoint, is thus always possible by means of trigonometric series with two arguments. In Part II of this report, we will return to this integration method.

In this Section 4, we will integrate eqs.(18) of Section 3 by a less general but more direct method which is also simpler from the viewpoint of numerical applications.

We will introduce the variables (24) of Section 3, which satisfy the equations /24

$$\frac{d\varphi'}{dt_1} = -2\sqrt{-1}\frac{dH}{d\psi'}, \quad \frac{d\psi'}{dt_1} = 2\sqrt{-1}\frac{dH}{d\varphi'},$$
$$\frac{d\varphi''}{dt_1} = -2\sqrt{-1}\frac{dH}{d\psi''}, \quad \frac{d\psi''}{dt_1} = 2\sqrt{-1}\frac{dH}{d\varphi''}. \tag{1}$$

The function H can be expanded in the form of

$$H = \sum_{m=0}^{\infty} \mu^m H^{(m)}, \tag{2}$$

where

$$H^{(m)} = \sum H^{m,\,\alpha'+\beta',\,\alpha''+\beta''}_{\alpha'-\beta',\,\alpha''-\beta''}\, \varphi'^{\alpha'} \psi'^{\beta'} \varphi''^{\alpha''} \psi''^{\beta''}. \tag{3}$$

Here, according to our statements in the preceding Section, the nonnegative whole numbers α', β', α'', β'' take only the values that satisfy the conditions

$$\begin{aligned} &\alpha' + \beta' + \alpha'' + \beta'' \geq 2, \quad \alpha'' + \beta'' = \text{even}, \\ 2m + 2 &= |\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' + 2k \\ &= |\alpha' - \beta' + \alpha'' - \beta''| + |\alpha' - \beta'| + |\alpha'' - \beta''| + 2\tilde{k}, \end{aligned} \tag{4}$$

where k and $\tilde{k}$ are nonnegative integers. According to eqs.(28) and (29) in Section 3, the function $H^{(m)}$, with respect to the variables e_0, φ', ψ', φ'', ψ'', is a real and even polynomial of the degree $2m + 2$.

Since the formula (32) of Section 3 is given, the function H exhibits a remarkable symmetry which is expressed by the identical relation

$$H(\varphi', \psi', \varphi'', \psi'') \equiv H(\psi', \varphi', \psi'', \varphi''). \tag{5}$$

The function $H^{(0)}$ has the following especially simple form:

$$H^{(0)} = -\frac{\nu_0'}{2}\varphi'\psi' - \frac{\nu_0''}{2}\varphi''\psi''.$$

In view of the conditions (4), the numbers α', β', α'', β'' assume only the following values in $H^{(1)}$: /25

	α'	β'	α''	β''
*	2	2	0	0
	0	2	2	0
*	1	1	1	1
	2	0	0	2
*	0	0	2	2
	2	1	0	0
	1	2	0	0
	0	1	2	0
	1	0	1	1
	0	1	1	1
	1	0	0	2
	2	0	0	0
*	1	1	0	0
	0	2	0	0
	0	0	2	0
*	0	0	1	1
	0	0	0	2

Here, we indicated by an asterisk any combinations such that $\alpha' - \beta' = \alpha'' - \beta''$. In all these combinations, we have $\alpha' - \beta' = \alpha'' - \beta'' = 0$.

We will now demonstrate that it is possible to formally satisfy eqs.(1) by expanding φ', ψ', φ'', ψ'' in powers of μ and in accordance with the multiples of the two arguments

$$w' = \nu' t_1 + \gamma', \quad w'' = \nu'' t_1 + \gamma''.$$

Here, we used γ' and γ'' for denoting two arbitrary constants and ν' and ν'' for denoting two still unknown quantities.

We then introduce two additional arbitrary constants ρ_0' and ρ_0''. We will also put /26

$$\begin{aligned} \varphi_0' &= \rho_0' e^{\sqrt{-1}\, w'}, \quad \psi_0' = \rho_0' e^{-\sqrt{-1}\, w'}, \\ \varphi_0'' &= \rho_0'' e^{\sqrt{-1}\, w''}, \quad \psi_0'' = \rho_0'' e^{-\sqrt{-1}\, w''}. \end{aligned} \tag{6}$$

It is convenient to consider φ', ψ', φ'', ψ'' as functions of the independent variables φ_0', ψ_0', φ_0'', ψ_0'' instead of the variable t_1. We will then have

$$\frac{1}{\sqrt{-1}}\frac{d}{dt_1} = \nu' D' + \nu'' D''$$

with the symbolic notations

$$D' = \varphi'_0 \frac{d}{d\varphi'_0} - \psi'_0 \frac{d}{d\psi'_0}, \quad D'' = \varphi''_0 \frac{d}{d\varphi''_0} - \psi''_0 \frac{d}{d\psi''_0}.$$

Thus, eqs.(1) can be written as .

$$\begin{aligned}
(-\nu' D' - \nu'' D'' + \nu'_0)\,\varphi' &= 2\frac{d(H - H^{(0)})}{d\psi'},\\
(\nu' D' + \nu'' D'' + \nu'_0)\,\psi' &= 2\frac{d(H - H^{(0)})}{d\varphi'},\\
(-\nu'' D'' - \nu' D' + \nu''_0)\,\varphi'' &= 2\frac{d(H - H^{(0)})}{d\psi''},\\
(\nu'' D'' + \nu' D' + \nu''_0)\,\psi'' &= 2\frac{d(H - H^{(0)})}{d\varphi''};
\end{aligned}$$

or, still better,

$$\begin{aligned}
-(\nu' D' + \nu'' D'')(\psi'_0 \varphi') - (\nu' - \nu'_0)\,\psi'_0 \varphi' &= 2\psi'_0 \frac{d(H - H^{(0)})}{d\psi'}\\
(\nu' D' + \nu'' D'')(\varphi'_0 \psi') - (\nu' - \nu'_0)\,\varphi'_0 \psi' &= 2\varphi'_0 \frac{d(H - H^{(0)})}{d\varphi'}\\
-(\nu'' D'' + \nu' D')(\psi''_0 \varphi'') - (\nu'' - \nu''_0)\,\psi''_0 \varphi'' &= 2\psi''_0 \frac{d(H - H^{(0)})}{d\psi''}\\
(\nu'' D'' + \nu' D')(\varphi''_0 \psi'') - (\nu'' - \nu''_0)\,\varphi''_0 \psi'' &= 2\varphi''_0 \frac{d(H - H^{(0)})}{d\varphi''}
\end{aligned} \qquad (7)$$

In these equations, we will introduce the expansions /27

$$\begin{aligned}
\varphi' &= \varphi'_0 + \mu\varphi'_1 + \mu^2\varphi'_2 + \cdots = \Sigma\,\mu^k \varphi'_k,\\
\psi' &= \psi'_0 + \mu\psi'_1 + \mu^2\psi'_2 + \cdots = \Sigma\,\mu^k \psi'_k,\\
\varphi'' &= \varphi''_0 + \mu\varphi''_1 + \mu^2\varphi''_2 + \cdots = \Sigma\,\mu^k \varphi''_k,\\
\psi'' &= \psi''_0 + \mu\psi''_1 + \mu^2\psi''_2 + \cdots = \Sigma\,\mu^k \psi''_k;\\
\nu' &= \nu'_0 + \mu\nu'_1 + \mu^2\nu'_2 + \cdots = \Sigma\,\mu^k \nu'_k,\\
\nu'' &= \nu''_0 + \mu\nu''_1 + \mu^2\nu''_2 + \cdots = \Sigma\,\mu^k \nu''_k.
\end{aligned} \qquad (8)$$

Next, we expand the two terms of eqs.(7) in powers of μ. By equating, in the two members of these equations, all coefficients of the same power of μ, we obtain a sequence of equations by means of which the various coefficients of the expansions (8) can be determined.

First, eqs.(7) are obviously satisfied for $\mu = 0$.

By equating the coefficients of μ, we obtain the following equations:

$$\begin{aligned}
\nu'_0(-D'+D'')(\psi'_0\varphi'_1) - \nu'_1\psi'_0\varphi'_0 &= 2\psi'_0\frac{dH^{(1)}}{d\psi'_0},\\
\nu'_0(\;\;D'-D'')(\varphi'_0\psi'_1) - \nu'_1\varphi'_0\psi'_0 &= 2\varphi'_0\frac{dH^{(1)}}{d\varphi'_0},\\
\nu''_0(-D''+D')(\psi''_0\varphi''_1) - \nu''_1\psi''_0\varphi''_0 &= 2\psi''_0\frac{dH^{(1)}}{d\psi''_0},\\
\nu''_0(\;\;D''-D')(\varphi''_0\psi''_1) - \nu''_1\varphi''_0\psi''_0 &= 2\varphi''_0\frac{dH^{(1)}}{d\varphi''_0}.
\end{aligned} \tag{9}$$

Let f be any expandable function, under the form

$$f = \Sigma C\varphi'^{\alpha'}_0\psi'^{\beta'}_0\varphi''^{\alpha''}_0\psi''^{\beta''}_0 = \Sigma C\mathfrak{M}. \tag{10}$$

(For abbreviation, we use $\mathfrak{M} = \varphi'^{\alpha'}_0\psi'^{\beta'}_0\varphi''^{\alpha''}_0\psi''^{\beta''}_0$.) The coefficients C may depend in an arbitrary manner on the two quantities $\varphi'_0\psi'_0 = \rho'^2_0$ and $\varphi''_0\psi''_0 = \rho''^2_0$. Thus, we can denote by

$$[f]$$

the ensemble of the terms where $\alpha' - \beta' = \alpha'' - \beta''$, and by /28

$$[[f]]$$

the ensemble of the terms where $\alpha' - \beta' = \alpha'' - \beta'' = 0$. Finally, we put

$$\{f\} = f - [f].$$

Obviously, f is a periodic function of the two arguments w' and w''; [f] is the portion of f which only includes the multiples of the argument $w' + w''$; finally [[f]] is the mean value of f, i.e., the term independent of w' and w''.

Since the conditions (4) or else the values of the Table on p.20 are given, it is obvious that

$$\begin{aligned}
[H^{(1)}] = [[H^{(1)}]] = H^{1,4,0}_{0,0}\varphi'^2_0\psi'^2_0 + H^{1,2,2}_{0,0}\varphi'_0\psi'_0\varphi''_0\psi''_0 + H^{1,0,4}_{0,0}\varphi''^2_0\psi''^2_0\\
+ H^{1,2,0}_{0,0}\varphi'_0\psi'_0 + H^{1,0,2}_{0,0}\varphi''_0\psi''_0.
\end{aligned} \tag{11}$$

Thus, eqs.(9) are satisfied by setting

$$\begin{aligned}
\nu'_1 &= -2\frac{d[H^{(1)}]}{\varphi'_0 d\psi'_0} = -2\frac{d[H^{(1)}]}{\psi'_0 d\varphi'_0}\\
&= -4H^{1,4,0}_{0,0}\varphi'_0\psi'_0 - 2H^{1,2,2}_{0,0}\varphi''_0\psi''_0 - 2H^{1,2,0}_{0,0},\\
\nu''_1 &= -2\frac{d[H^{(1)}]}{\varphi''_0 d\psi''_0} = -2\frac{d[H^{(1)}]}{\psi''_0 d\varphi''_0}\\
&= -4H^{1,0,4}_{0,0}\varphi''_0\psi''_0 - 2H^{1,2,2}_{0,0}\varphi'_0\psi'_0 - 2H^{1,0,2}_{0,0};
\end{aligned} \tag{12}$$

$$\Phi'[i] = 2\left(\frac{d^2H^{(1)}}{d\psi'_0 d\varphi'_0}\frac{[\psi'_0\varphi'_i]}{\psi'_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\psi'_0}\frac{[\varphi'_0\psi'_i]}{\varphi'_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\varphi''_0}\frac{[\psi''_0\varphi''_i]}{\psi''_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\psi''_0}\frac{[\varphi''_0\psi''_i]}{\varphi''_0}\right),$$

$$\Phi'\{i\} = 2\left(\frac{d^2H^{(1)}}{d\psi'_0 d\varphi'_0}\frac{\{\psi'_0\varphi'_i\}}{\psi'_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\psi'_0}\frac{\{\varphi'_0\psi'_i\}}{\varphi'_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\varphi''_0}\frac{\{\psi''_0\varphi''_i\}}{\psi''_0} + \frac{d^2H^{(1)}}{d\psi'_0 d\psi''_0}\frac{\{\varphi''_0\psi''_i\}}{\varphi''_0}\right) \tag{12'}$$

In the Σ' there are no terms where the divisor $\alpha' - \beta' - \alpha'' + \beta''$ is canceled. /29
These terms have been eliminated by the selection of ν'_1 and ν''_1.

Since $H^{(1)}$ is an even polynomial of the fourth degree in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0, it is obvious that ν'_1 and ν''_1 are linear in e_0^2, $\rho_0'^2$, $\rho_0''^2$ and that the functions (12') are polynomials of the fourth degree in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0 having only terms of even dimensions.

Equations (12') give only a part of the unknown functions. The quantities $[\psi'_0\varphi'_1]$, $[\varphi'_0\psi'_1]$, $[\psi''_0\varphi''_1]$, $[\varphi''_0\psi''_1]$ still remain unknown.

In what follows, we will often find it advantageous to make use of the notations

$$\{\psi'_0\varphi'_i\} = \frac{2}{\nu'_0}\sum{}' \frac{\beta' H^{1,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''}}{-\alpha'+\beta'+\alpha''-\beta''}\mathfrak{M},$$

$$\{\varphi'_0\psi'_i\} = \frac{2}{\nu'_0}\sum{}' \frac{\alpha' H^{1,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''}}{\alpha'-\beta'-\alpha''+\beta''}\mathfrak{M},$$

$$\{\psi''_0\varphi''_i\} = \frac{2}{\nu''_0}\sum{}' \frac{\beta'' H^{1,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''}}{\alpha'-\beta'-\alpha''+\beta''}\mathfrak{M}, \tag{13}$$

$$\{\varphi''_0\psi''_i\} = \frac{2}{\nu''_0}\sum{}' \frac{\alpha'' H^{1,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''}}{-\alpha'+\beta'+\alpha''-\beta''}\mathfrak{M}.$$

and of other analogous expressions $\Psi'[i]$ and $\Psi'\{i\}$ which are obtained from the expressions (13) by writing everywhere φ instead of ψ and vice versa. Finally, this will result in four expressions $\Phi''[i]$, $\Phi\{i\}$, $\Psi''[i]$, $\Psi''\{i\}$ by permitting the indices ' and " in the above-defined four expressions.

Thus, let us now compare the coefficients of μ^2 in the two members of each of the equations of the system (7). This will yield four equations, of which the first reads:

$$\nu'_0(-D'+D'')(\psi'_0\varphi'_2) - (\nu'_1 D' + \nu''_1 D'' + \nu'_1)[\psi'_0\varphi'_1] - \nu'_2\psi'_0\varphi'_0 - \psi'_0\Phi'[1] = \psi'_0 A'_2. \tag{14}$$

We have set here

$$\psi'_0 A'_2 = (\quad \nu'_1 D' + \nu''_1 D'' + \nu'_1)\{\psi'_0 \varphi'_1\} + \psi'_0 \Phi'\{1\} + 2\psi'_0 \frac{dH^{(2)}}{d\psi'_0},$$ /30

$$\varphi'_0 B'_2 = (-\nu'_1 D' - \nu''_1 D'' + \nu'_1)\{\varphi'_0 \psi'_1\} + \varphi'_0 \Psi'\{1\} + 2\varphi'_0 \frac{dH^{(2)}}{d\varphi'_0},$$

$$\psi''_0 A''_2 = (\quad \nu''_1 D'' + \nu'_1 D' + \nu''_1)\{\psi''_0 \varphi''_1\} + \psi''_0 \Phi''\{1\} + 2\psi''_0 \frac{dH^{(2)}}{d\psi''_0}, \tag{15}$$

$$\varphi''_0 B''_2 = (-\nu''_1 D'' - \nu'_1 D' + \nu''_1)\{\varphi''_0 \psi''_1\} + \varphi''_0 \Psi''\{1\} + 2\varphi''_0 \frac{dH^{(2)}}{d\varphi''_0},$$

which are entirely known functions.

The second equation (14) is obtained from the first equation by writing everywhere ψ, Ψ, B instead of φ, Φ, A and vice versa and by also changing the signs of D' and D''. The two last equations are obtained by permuting the indices $'$ and $''$ in writing the two first equations.

Primarily, it is necessary to determine the functions $[\psi'_0, \varphi'_1]$, $[\varphi'_0, \psi'_1]$, $[\psi''_0, \varphi''_1]$, $[\varphi''_0, \psi''_1]$ as well as the quantities ν'_2 and ν''_2 in such a manner that all the terms where $\alpha' - \beta' - \alpha'' + \beta'' = 0$ will vanish from the equations (14). This will yield four equations of which we give the first:

$$\begin{aligned}
&-(\nu'_1 D' + \nu''_1 D'' + \nu'_1)[\psi'_0 \varphi'_1] - \nu'_2 \psi'_0 \varphi'_0 \\
&-2\frac{d^2[H^{(1)}]}{d\psi'_0 d\varphi'_0}[\psi'_0 \varphi'_1] - 2\frac{\psi'_0}{\varphi'_0}\frac{d^2[H^{(1)}]}{d\psi'_0 d\psi'_0}[\varphi'_0 \psi'_1] \\
&-2\frac{\psi'_0}{\psi''_0}\frac{d^2[H^{(1)}]}{d\psi'_0 d\varphi''_0}[\psi''_0 \varphi''_1] - 2\frac{\psi'_0}{\varphi''_0}\frac{d^2[H^{(1)}]}{d\psi'_0 d\psi''_0}[\varphi''_0 \psi''_1] = [\psi'_0 A'_2].
\end{aligned}$$

Since eqs.(11) and (12) are given, it is easy to demonstrate that this equation will take the form

$$\begin{aligned}
&-(\nu'_1 D' + \nu''_1 D'')[\psi'_0 \varphi'_1] - (4H^{1,4,0}_{0,0}[\psi'_0 \varphi'_1 + \varphi'_0 \psi'_1] \\
&\qquad + 2H^{1,2,2}_{0,0}[\psi''_0 \varphi''_1 + \varphi''_0 \psi''_1] + \nu'_2)\psi'_0 \varphi'_0 = [\psi'_0 A'_2].
\end{aligned} \tag{16}$$

The first member of the second equation (16), not written down, is obtained by changing the sign of D' and D'' and by permuting $[\psi'_0 \varphi'_1]$ and $[\varphi'_0 \psi'_1]$ in the first member of eq.(16). Finally, the first members of the two last equations are /31 obtained from the first members of the two first equations by permuting the indices $'$ and $''$ and by writing $H^{1,0,4}_{0,0}$ instead of $H^{1,4,0}_{0,0}$.

In view of the symmetry of the function H, which is expressed by the identity (5), it is obvious that the functions A'_2 and B'_2 as well as A''_2 and B''_2 are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0.

Let us now form the difference of the two first equations as well as of the last two equations of the system (16). This will yield

$$-(\nu'_1 D' + \nu''_1 D'')[\psi'_0 \varphi'_1 + \varphi'_0 \psi'_1] = [\psi'_0 A'_2] - [\varphi'_0 B'_2], \tag{17}$$

$$-(\nu_1'' D'' + \nu_1' D')[\psi_0''\varphi_1'' + \varphi_0''\psi_1''] = [\psi_0'' A_2''] - [\varphi_0'' B_2''].$$

We now make the statement that the functions $[\psi_0' A_2']$, $[\varphi_0' B_2']$, $[\psi_0'' A_2'']$, $[\varphi_0'' B_2'']$ are polynomials in $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$ such that the second terms of eqs.(17) will cancel out.

It is sufficient to examine the function

$$[\psi_0' A_2'] = \left[\psi_0' \Phi'\{1\} + 2\psi_0' \frac{dH^{(2)}}{d\psi_0'}\right].$$

The function $\psi_0' \Phi'\{1\}$, according to the second formula in eq.(13), is composed of four terms. Each of these terms includes, as factor, one of the functions (12'). The exponents α', β', α'', β'', of the monomials $\mathfrak{M}$ of these functions satisfy the third relation of eq.(4) for m = 1, i.e., satisfy the relation

$$4 = |\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' + 2k_1 \quad (k_1 > 0).$$

For the exponents α', β', α'', β'' of the monomials $\mathfrak{M}$ of the other factor (for example, of the factor $\left.\frac{\psi_0'}{\varphi_0'}\frac{d^2H^{(1)}}{d\psi_0' d\varphi_0'}\right)$), we obviously have

$$2 = |\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' + 2k_2 \quad (k_2 \geq 0).$$

For the exponents α', β', α'', β'' of the monomials $\mathfrak{M}$ of the product of the two factors, we thus will obtain

$$6 = |\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' + 2k \quad (k \geq 0)$$

or else, since the α', β', α'', β'' of this latter formula are positive, /32

$$6 = |\alpha' - \beta' + \alpha'' - \beta''| + |\alpha' - \beta'| + |\alpha'' - \beta''| + 2\tilde{k} \quad (\tilde{k} \geq 0). \tag{18}$$

Let us now consider the logarithmic derivative of the function $H^{(2)}$. The exponents α', β', α'', β'' of $H^{(2)}$ and of its logarithmic derivatives satisfy the relations (4) for m = 2, i.e., also satisfy the relation (18).

In the function $[\psi_0' A_2']$ where $\alpha' - \beta' = \alpha'' - \beta''$ = even, we thus have

$$6 = 4|\alpha' - \beta'| + 2\tilde{k} \qquad (\tilde{k} > 0)$$

and, consequently,

$$\alpha' - \beta' = \alpha'' - \beta'' = 0.$$

This is the same for the functions $[\varphi_0' B_2']$, $[\psi_0'' A_2'']$, $[\varphi_0'' B_2'']$.

Thus the second members of eqs.(16) are polynomials in $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$.

Since, in addition, the functions $[\psi_0' A_2']$ and $[\varphi_0' B_2']$ as well as $[\psi_0'' A_2'']$ and $[\varphi_0'' B_2'']$ are permuted on permuting φ_0' and ψ_0' simultaneously with φ_0'' and ψ_0'', it is obvious that the second terms of eqs.([illegible]) are canceled.

Let us then assume that the unknown functions $[\psi_0'\varphi_1' + \varphi_0'\psi_1']$ and $[\psi_0''\varphi_1'' + \varphi_0''\psi_1'']$ can be expanded in the form of eq.(10). Then, the exponents in the monomials $\mathfrak{M}$ of these functions will satisfy the equations

$$\nu_1'(\alpha'-\beta') + \nu_1''(\alpha''-\beta'') = 0,$$
$$(\alpha'-\beta') - (\alpha''-\beta'') = 0,$$

from which it follows that $\alpha' - \beta' = \alpha'' - \beta'' = 0$. Thus, these two functions depend only on the products $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$, i.e., they are constants. We can equate these to zero without restricting the generality since ρ_0' and ρ_0'' are arbitrary quantities.

Accordingly, it is obvious that we can satisfy eqs.(16) by putting /33

$$\nu_2' = -\frac{[[\psi_0' A_2']]}{\varphi_0'\psi_0'}, \quad \nu_2'' = -\frac{[[\psi_0'' A_2'']]}{\varphi_0''\psi_0''},$$
$$[\psi_0'\varphi_1'] = [\varphi_0'\psi_1'] = [\psi_0''\varphi_1''] = [\varphi_0''\psi_1''] = 0.$$

Evidently, the quantities $[[\psi_0'A_2']]$ and $[[\psi_0''A_2'']]$ are polynomials in e_0^2, $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$. The degree of these polynomials is at most 3, according to what we know of the degree of $H^{(1)}$ and $H^{(2)}$. In addition, $[[\psi_0'A_2']]$ is divisible by $\varphi_0'\psi_0'$ and $[[\psi_0''A_2'']]$ by $\varphi_0''\psi_0''$, since $\psi_0'A_2'$ is divisible by ψ_0', and $\psi_0''A_2''$ is divisible by ψ_0''. Thus, ν_2' and ν_2'' are polynomials of the second degree in e_0^2, $\varphi_0'\psi_0'$, $\varphi_0''\psi_0''$, i.e., polynomials of the second degree in e_0^2, $\rho_0'^2$, and $\rho_0''^2$.

We then will return to eqs.(14). All the terms, except the first, in each equation of this system are now known. The known terms form even polynomials of the sixth degree in e_0, φ_0', ψ_0', φ_0'', ψ_0''; in these polynomials, the exponents α', β', α'', β'' of the monomials $\mathfrak{M}$ satisfy the inequality

$$\alpha' - \beta' - \alpha'' + \beta'' \neq 0.$$

Because of eqs.(14), we give the functions $\{\psi_0'\varphi_2'\}$, $\{\varphi_0'\psi_2'\}$, $\{\psi_0''\varphi_2''\}$, $\{\varphi_0''\psi_2''\}$ as even polynomials of the sixth degree in e_0, φ_0', ψ_0', φ_0'', ψ_0''.

It has been demonstrated that the wholly known terms of the first two equations of the system (14) as well as the wholly known terms of the two last equations are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''. It must be concluded from this that the functions $\{\psi_0'\varphi_2'\}$ and $\{\varphi_0'\psi_2'\}$ as well as $\{\psi_0''\varphi_2''\}$ and $\{\varphi_0''\psi_2''\}$ are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''. Moreover, $\{\psi_0'\varphi_2'\}$ is divisible by ψ_0', and $\{\varphi_0'\psi_2'\}$ is divisible by φ_0', $\{\psi_0''\varphi_2''\}$ by ψ_0'', and $\{\varphi_0''\psi_2''\}$ by φ_0''.

Let us now compare the coefficients of μ^3 in the two members of each of the equations in system (7). This will yield four equations, of which it is sufficient to write the first: /34

$$\nu_0'(-D'+D'')(\psi_0'\varphi_3') - (\nu_1'D' + \nu_1''D'' + \nu_1')[\psi_0'\varphi_2'] - \nu_3'\psi_0'\varphi_0' - \psi_0'\Phi'[2] = \psi_0'A_3'. \qquad (19)$$

Here, we denoted by A'_3 (B'_3, A''_3, B''_3 in the three equations not given here) the wholly known polynomials which are odd and have the seventh degree in a_0, φ'_0, ψ'_0, φ''_0, ψ''_0. The quantities A'_3 and B'_3 as well as A''_3 and B''_3 are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0.

Primarily, we must select the functions $[\psi'_0\varphi'_2]$, $[\varphi'_0\psi'_2]$, $[\psi''_0\varphi''_2]$ and $[\varphi''_0\psi''_2]$ as well as the quantities ν'_3 and ν''_3 such that all the terms where $\alpha' - \beta' - \alpha'' + \beta'' = 0$ will vanish from eq.(19). This will yield four equations, analogous to eq.(16). It is sufficient to write only the first:

$$-(\nu'_1 D' + \nu''_1 D'')[\psi'_0\varphi'_2] - (4H^{1,4,0}_{0,0}[\psi'_0\varphi'_2 + \varphi'_0\psi'_2] + 2H^{1,2,2}_{0,0}[\psi''_0\varphi''_2 + \varphi''_0\psi''_2] + \nu'_3)\,\psi'_0\varphi'_0 = [\psi'_0 A'_3]. \tag{20}$$

The first term in the second equation (not given here) is obtained by changing the sign of D' and D'' in the first member of eq.(20) and by there permuting $[\psi'_0\varphi'_2]$ and $[\varphi'_0\psi'_2]$. Next, the first members of the two last equations are obtained from the corresponding members of the first two equations by permuting the superscripts $'$ and $''$ as well as the coefficients $H^{1,4,0}_{0,0}$ and $H^{1,0,4}_{0,0}$.

Let us denote by

$$\Sigma A'\mathfrak{M},\quad \Sigma B'\mathfrak{M},\quad \Sigma A''\mathfrak{M},\quad \Sigma B''\mathfrak{M}$$

the expansions of the four functions

$$[\psi'_0 A'_3],\quad [\varphi'_0 B'_3],\quad [\psi''_0 A''_3],\quad [\varphi''_0 B''_3]$$

in powers of φ'_0, ψ'_0, φ''_0, ψ''_0. The exponents α', β', α'', β'' of the monomials $\mathfrak{M}$ then satisfy the condition

$$\alpha' - \beta' - \alpha'' + \beta'' = 0.$$

Let us now form the difference of the two first equations of the system (20) and also the difference of the two last equations. This will yield

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$$\begin{aligned} -(\nu'_1 D' + \nu''_1 D'')[\psi'_0\varphi'_2 + \varphi'_0\psi'_2] &= [\psi'_0 A'_3] - [\varphi'_0 B'_3], \\ -(\nu''_1 D'' + \nu'_1 D')[\psi''_0\varphi''_2 + \varphi''_0\psi''_2] &= [\psi''_0 A''_3] - [\varphi''_0 B''_3]. \end{aligned} \tag{21}$$

The second members of these equations change their signs on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0. From this it follows that we have no terms where $\alpha' - \beta' = \alpha'' - \beta'' = 0$.

Thus, the following conclusion can be drawn from eqs.(21):

$$[\psi'_0\varphi'_2 + \varphi'_0\psi'_2] = \frac{1}{\nu'_1 + \nu''_1}\sum \frac{A' - B'}{-\alpha' + \beta'}\,\mathfrak{M}, \tag{22}$$

$$[\psi_0''\varphi_2'' + \varphi_0''\psi_2''] = \frac{1}{\nu_1' + \nu_1''}\sum \frac{A''-B''}{-\alpha''+\beta''}\mathfrak{M}.$$

We could have added to the second members of these equations any functions that depend only on $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$, but such functions can be omitted without restricting the generality since ρ_0' and ρ_0'' are already arbitrary quantities.

Let us now put in eqs.(20):

$$\nu_2' = -\frac{[[\psi_0' A_2']]}{\varphi_0'\psi_0'}, \quad \nu_2'' = -\frac{[[\psi_0'' A_2'']]}{\varphi_0''\psi_0''}.$$

These quantities are polynomials of the third degree in e_0^2, $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$.

Now, eqs.(20) can be integrated. On setting, for abbreviation,

$$\begin{aligned}
\chi' = {} & 4H_{0,0}^{1,4,0}\sum \frac{A'-B'}{(\alpha'-\beta')^2}\mathfrak{M} \\
& + 2H_{0,0}^{1,2,2}\sum \frac{A''-B''}{(\alpha''-\beta'')^2}\mathfrak{M}, \\
\chi'' = {} & 4H_{0,0}^{1,0,4}\sum \frac{A''-B''}{(\alpha''-\beta'')^2}\mathfrak{M} \\
& + 2H_{0,0}^{1,2,2}\sum \frac{A'-B'}{(\alpha'-\beta')^2}\mathfrak{M},
\end{aligned}$$

we find /36

$$\begin{aligned}
[\psi_0'\varphi_2'] &= \frac{\varphi_0'\psi_0'}{(\nu_1'+\nu_1'')^2}\chi' + \frac{1}{\nu_1'+\nu_1''}\sum' \frac{A'}{-\alpha'+\beta'}\mathfrak{M}, \\
[\varphi_0'\psi_2'] &= -\frac{\varphi_0'\psi_0'}{(\nu_1'+\nu_1'')^2}\chi' - \frac{1}{\nu_1'+\nu_1''}\sum' \frac{B'}{-\alpha'+\beta'}\mathfrak{M}, \\
[\psi_0''\varphi_2''] &= \frac{\varphi_0''\psi_0''}{(\nu_1'+\nu_1'')^2}\chi'' + \frac{1}{\nu_1'+\nu_1''}\sum' \frac{A''}{-\alpha''+\beta''}\mathfrak{M}, \\
[\varphi_0''\psi_2''] &= -\frac{\varphi_0''\psi_0''}{(\nu_1'+\nu_1'')^2}\chi'' - \frac{1}{\nu_1'+\nu_1''}\sum' \frac{B''}{-\alpha''+\beta''}\mathfrak{M}.
\end{aligned} \tag{23}$$

In the sums Σ', the terms where $\alpha' - \beta' = \alpha'' - \beta'' = 0$ must be excluded.

According to the above statements on the second members of eqs.(20), it is obvious that χ' and χ'' change their signs and that $[\psi_0'\varphi_2']$ and $[\varphi_0'\psi_2']$ as well as $[\psi_0''\varphi_2'']$ and $[\varphi_0''\psi_2'']$ are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''. From this it follows also that $[\psi_0'\varphi_2']$ is divisible by ψ_0'; $[\varphi_0'\psi_2']$ by φ_0'; $[\psi_0''\varphi_2'']$ by ψ_0''; and $[\varphi_0''\psi_2'']$ by φ_0''.

Obviously, we can throw the functions (23) into a rational form with the denominator $(\nu_1' + \nu_1'')^2$, since the numerators are even polynomials of the tenth degree in e_0, φ_0', ψ_0', φ_0'', ψ_0''.

It is now possible to integrate eqs.(19) and to derive from these the expressions for $\{\psi_0'\varphi_3'\}$, $\{\varphi_0'\psi_3'\}$, $\{\psi_0''\varphi_3''\}$, $\{\varphi_0''\psi_3''\}$. These functions, respectively, are divisible by ψ_0, φ_0', ψ_0', φ_0''. The two first expressions as well as the two last expressions will be permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''. Finally, these four functions are fractions whose denominator is $(\nu_1' + \nu_1'')^2$ whereas the numerators are even polynomials of the twelfth degree in e_0, φ_0', ψ_0', φ_0'', ψ_0''.

Evidently, it is possible to continue in this manner and to successively terminate the various coefficients of the expansions (8).

It is obvious that ν_k' and ν_k'' are rational in e_0^2, $\varphi_0'\psi_0'$, $\varphi_0''\psi_0''$; that φ_k', ψ_k' φ_0'', ψ_k'' are rational and odd in e_0, φ_0', ψ_0', φ_0'', ψ_0''; and that the denominators of all these rational functions are powers of the quantity $\nu_1' + \nu_1''$. /37

It is also clear that φ_k' and ψ_k' as well as φ_k'' and ψ_k'' are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''.

Section 5.

Below, we will demonstrate several general propositions on the quantities

$$\nu_k', \quad \nu_k'' \tag{1}$$

and on the functions

$$\{\psi_0'\varphi_k'\}, \ \{\varphi_0'\psi_k'\}, \ \{\psi_0''\varphi_k''\}, \ \{\varphi_0''\psi_k''\}, \tag{2}$$

$$[\psi_0'\varphi_k'], \ [\varphi_0'\psi_k'], \ [\psi_0''\varphi_k''], \ [\varphi_0''\psi_k''], \tag{3}$$

$$\frac{\{\psi_0'\varphi_k'\}}{\psi_0'}, \ \frac{\{\varphi_0'\psi_k'\}}{\varphi_0'}, \ \frac{\{\psi_0''\varphi_k''\}}{\psi_0''}, \ \frac{\{\varphi_0''\psi_k''\}}{\varphi_0''}, \tag{4}$$

$$\frac{[\psi_0'\varphi_k']}{\psi_0'}, \ \frac{[\varphi_0'\psi_k']}{\varphi_0'}, \ \frac{[\psi_0''\varphi_k'']}{\psi_0''}, \ \frac{[\varphi_0''\psi_k'']}{\varphi_0''} \tag{5}$$

defined in the preceding Section.

Let us designate as the degree of a rational function the difference in the degrees of the numerator and the denominator.

The first proposition is as follows:

<u>Theorem 1.</u> The quantities (1) which are rational in e_0^2, $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$ (i.e., in e_0^2, $\rho_0'^2$ and $\rho_0''^2$) are at most of the degree 2k with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0''. For k = 1, 2, 3, no denominator exists. For k > 3, the denominator has the form $(\nu_1' + \nu_1'')^s$ where $s \le 2k - 6$. The functions (4) and (5) which are

rational in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0 are at most of the degree $2k + 1$. The denominator has the form $(\nu'_1 + \nu''_1)^s$ where $s \leq 2k - 4$ for the functions (4) and $s \leq$ $\leq 2k - 2$ for the functions (5).

The demonstration is performed by induction from i to $i + 1$. Let us, therefore, assume that the theorem is true in the case of the quantities (1) and (4) for $k = 1, 2, \ldots i$ and in the case of the functions (5) for $k = 1, 2, \ldots i - 1$; let us demonstrate that this theorem still remains true after increasing i by unity.

A comparison, in eqs.(7) of Section 4, of the coefficients of μ^{i+1} in the two members, will yield four equations of which only the first need be given: /38

$$\nu'_0(-D' + D'')(\psi'_0\varphi'_{i+1}) - (\nu'_1 D' + \nu''_1 D'' + \nu'_1)[\psi'_0\varphi'_i] - \nu'_{i+1}\psi'_0\varphi'_0 - \psi'_0\Phi'[i] = \psi'_0 A'_{i+1}. \tag{6}$$

We will first demonstrate that the rational function A'_{i+1} is at most of the degree $2i + 3$ in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0 and that the denominator includes $\nu'_1 + \nu''_1$ at most to the power $2i - 4$.

In fact, $\psi'_0 A'_{i+1}$ primarily encompasses the terms

$$(\nu'_1 D' + \nu''_1 D'' + \nu'_1)\{\psi'_0\varphi'_i\} + (\nu'_2 D' + \nu''_2 D'' + \nu'_2)(\psi'_0\varphi'_{i-1}) + \cdots + (\nu'_i D' + \nu''_i D'' + \nu'_i)(\psi'_0\varphi'_1). \tag{7}$$

According to the assumptions made, these terms are at most of the degree $2i + 4$ and, in the denominator, include $\nu'_1 + \nu''_1$ at most to the power $2i - 4$.

Then, the following quantity will be encountered in $\psi'_0 A'_{i+1}$:

$$\psi'_0\Phi'\{i\} \tag{8}$$

[see the second formula (13) in Section 4]. At most, this quantity is of the degree $2i + 4$ and the denominator includes the quantity $\nu'_1 + \nu''_1$ at most to the power $2i - 4$.

Finally, $\psi'_0 A'_{i+1}$ contains terms of the form (to within a numerical factor)

$$\varphi_1'^{m'_1}\psi_1'^{n'_1}\varphi_1''^{m''_1}\psi_1''^{n''_1}\cdots\varphi_{i-1}'^{m'_{i-1}}\psi_{i-1}'^{n'_{i-1}}\varphi_{i-1}''^{m''_{i-1}}\psi_{i-1}''^{n''_{i-1}}. \tag{9}$$
$$\psi'_0\cdot\frac{d^{1+\Sigma(m'+n'+m''+n'')}H^{(m)}}{d\psi'_0(d\varphi'_0)^{\Sigma m'}(d\psi'_0)^{\Sigma n'}(d\varphi''_0)^{\Sigma m''}(d\psi''_0)^{\Sigma n''}}$$

with the condition

$$\sum_{k=1}^{i-1} k(m'_k + n'_k + m''_k + n''_k) + m = i + 1 \tag{10}$$

and also the condition /39

$$\Sigma(m'+n'+m''+n'')+m>2, \tag{11}$$

provided that $i > 1$, which we assume here. The degree of the term (9) will be at most

$$\sum_{k=1}^{i-1}(2k+1)(m'_k+n'_k+m''_k+n''_k)+ \\ +2m+2-\Sigma(m'+n'+m''+n'')=2i+4.$$

In addition, this term has the quantity $\nu_1' + \nu_1''$ in its denominator, raised at most to the power

$$\sum_{k=1}^{i-1}(2k-2)(m'_k+n'_k+m''_k+n''_k)= \\ =2(i+1-m)-2\Sigma(m'+n'+m''+n'')\leq 2i-4.$$

Thus, the function A'_{i+1} is at most of the degree $2i + 3$ and, in its denominator, the quantity $\nu_1' + \nu_1''$ enters at most at the power $2i - 4$.

After this and since (by Section 4) the method of formation of the quantities (1) for $k = i + 1$, of the functions (5) for $k = i$, and of the functions (4) for $k = i + 1$ are given by means of eqs.(6), it is obvious that these quantities and these functions have exactly the properties enumerated in the statement of the theorem. Thus, assuming that the theorem is true in the case of the quantities (1) and (4) for $k = 1, 2, \ldots i$ as well as in the case of the functions (5) for $k = 1, 2, \ldots i - 1$, we have demonstrated that the theorem remains true also on increasing i by unity. However, according to Section 4, our suppositions are exact up to $i = 3$. Thus, theorem 1 is proved.

It is obvious that the functions (2), ... (5) can be given the following form:

$$\Sigma C\varphi_0'^{\alpha'}\psi_0'^{\beta'}\varphi_0''^{\alpha''}\psi_0''^{\beta''}=\Sigma C\mathfrak{M}, \tag{12}$$

where C are rational functions in $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$. For abbreviation, we will set, in a given expansion of the form of eq.(12), /40

$$j'=\alpha'-\beta',\quad j''=\alpha''-\beta'',$$
$$S=|j'+j''|,\quad T=|j'|+|j''|.$$

Consider any monomial in φ_0', ψ_0', φ_0'', ψ_0''. It is evident that, for such a monomial, the values of S and of T cannot exceed the degree of the monomial.

Let us study particularly the polynomials $H^{(m)}$ defined by eq.(3) in Section 4. For these, we have the frequently used relation:

$$S+T\leq 2m+2.$$

Since we always have $S \leq T$, the following expression is obtained for these polynomials $H^{(m)}$ and for their logarithmic derivatives:

$$S \leq m + 1, \quad T \leq 2m + 2.$$

In view of this, we will demonstrate the following proposition:

Theorem 2. For the functions (2) and (3), derived in the form of eq.(12), we will have

$$S \leq 2k. \tag{13}$$

For the functions (3), we will have $S = 2k$ provided that k is even and then only in the terms where

$$j' = j'' = \pm k. \tag{14}$$

For a function (2) we will have $S = 2k$ if and only if k is odd and then only for the combination (j', j'') written in the Table (15) below the investigated function:

$$\begin{array}{cccc} \{\psi_0'\varphi_k'\} & \{\varphi_0'\psi_k'\} & \{\psi_0''\varphi_k''\} & \{\varphi_0''\psi_k''\} \\ (-k-1, -k+1), & (k+1, k-1), & (-k+1, -k-1), & (k-1, k+1). \end{array} \tag{15}$$

The proof of theorem 2 is conducted by induction from i to $i + 1$. Let us assume that the proposition is true for $k = 1, 2, \dots i$ in the case of the functions (2), and for $k = 1, 2, \dots i - 1$ in the case of the functions (3). /41 We make the statement that the proposition remains true also on increasing i by unity.

To demonstrate this, it is necessary to investigate the function $\psi_0' A_{i+1}'$ which appears in the second term of eq.(6). We state that, in the monomials $\mathfrak{M}$ of this function, we have $S \leq 2i + 1$. In fact, the function $\psi_0' A_{i+1}'$ is composed of the parts (7) and (8) and of a sequence of terms of the form (9), multiplied by numerical coefficients.

For the part (7), we have $S \leq 2i$ in accordance with our suppositions.

The term of the formula (9) can also be written as follows (to within a factor which depends on $\varphi_0'\psi_0'$ and $\varphi_0''\psi_0''$):

$$\begin{gathered} (\psi_0'\varphi_1')^{m_1'}(\varphi_0'\psi_1')^{n_1'}(\psi_0''\varphi_1'')^{m_1''}(\varphi_0''\psi_1'')^{n_1''}\dots \\ (\psi_0'\varphi_{i-1}')^{m_{i-1}'}(\varphi_0'\psi_{i-1}')^{n_{i-1}'}(\psi_0''\varphi_{i-1}'')^{m_{i-1}''}(\varphi_0''\psi_{i-1}'')^{n_{i-1}''} \\ \psi_0'\varphi_0'^{\Sigma m'}\psi_0'^{\Sigma n'}\varphi_0''^{\Sigma m''}\psi_0''^{\Sigma n''}\frac{d^{1+\Sigma(m'+n'+m''+n'')}H^{(m)}}{d\psi_0'(d\varphi_0')^{\Sigma m'}(d\psi_0')^{\Sigma n'}(d\varphi_0'')^{\Sigma m''}(d\psi_0'')^{\Sigma n''}}. \end{gathered} \tag{16}$$

For this term, in accordance with our assumptions and in view of the condition (10), we have

$$S < \sum_{k=1}^{i-1} 2k\,(m'_k + n'_k + m''_k + n''_k) + m + 1 =$$
$$= 2i + 3 - m \le 2i + 1, \quad \text{si } m > 1.$$

If $m = 1$ in eq.(16), the order of the derivative of $H^{(1)}$ will be at least 3. Thus, in the polynomial which then appears in the second line of the expression (16), no terms of the second degree are present. For the terms of the third degree, we have $S = 1$ and, for the terms of the fourth degree, $S = 0$ (see the Table on p.20). Thus for $m = 1$, we will have in eq.(16):

$$S < \sum_{k=1}^{i-1} 2k\,(m'_k + n'_k + m''_k + n''_k) + 1 = 2i + 1.$$

Finally, let us investigate the part $\psi'_0\Phi'\{i\}$ of the function $\psi'_0 A'_{i+1}$. /42
According to the second formula (13) in Section 4, this part is composed of four terms, each of which is the product of two factors. For one of the factors which is one of the functions (2) for $k = i$, we have $S \le 2i$. For the other factor, as for $H^{(1)}$, we have $S \le 2$. Thus, in the function (8), we always have $S \le 2i + 2$. To have there $S = 2i + 2$, it would be necessary to take, in $H^{(1)}$, the term where $S = 2$ and $\beta' > 0$, i.e., the term $H^{1,2,0}_{-2,0}\psi'^2_0$ and, on the other hand, in $\{\varphi'_0\psi'_i\}$, the terms were $j' + j'' = -2i$. However, according to our assumptions, there are no such terms in $\{\varphi'_0\psi'_i\}$. [See the second column of Table (15) for $k = i$.] Thus, we have $S \le 2i + 1$ in the function (8) and also in the function $\psi'_0 A'_{i+1}$.

In view of this fact, we will have $S \le 2i$ in the function $[\psi'_0 A'_{i+1}]$ where S always is even.

Next, in accordance with the method of formation of the functions $[\psi'_0\varphi'_i]$ and $\{\psi'_0\varphi'_{i+1}\}$, it is obvious that we will have $S \le 2i$ in the first of these functions and then $S \le 2i + 2$ in the second function.

In the function $[\psi'_0\varphi'_i]$, the quantity S is divisible by 4. Thus, we will there have $S = 2i$ if and only if i is even and then only in the cases in which $j' = j'' = \pm i$.

In the function $\{\psi'_0\varphi'_{i+1}\}$, the terms where $S = 2i + 2$ can originate only in the part

$$\frac{4H^{1,2,0}_{-2,0}}{\varphi'_0\psi'_0}\psi'^2_0[\varphi'_0\psi'_i] \qquad (17)$$

of the function $\psi'_0\Phi'[i]$ in eq.(6). It is even sufficient to retain, in the function $[\varphi'_0\psi'_i]$ of eq.(17), only the term where $j' = j'' = -i$. However, such a term will exist only if i is even. Thus, we will have $S = 2i + 2$ in the function $\{\psi'_0\varphi'_{i+1}\}$ if and only if $i + 1$ is odd and thus only for $j' = -(i + 1) - 1$ and $j'' = -(i + 1) + 1$.

Analogous results can be obtained by permuting, in the preceding demonstration, the symbols φ and ψ or the superscripts $'$ and $''$.

Thus, assuming that the theorem 2 is true in the case of the functions (2) for $k = 1, 2, \dots i$ and in the case of the functions (3) for $k = 1, 2, \dots i - 1$, we will have demonstrated that the theorem remains true also on increasing i by unity. Moreover, our assumption is exact for $i = 2$. Thus, the theorem is proved. /43

It is now easy to demonstrate the following proposition:

<u>Theorem 3</u>: For the functions (4) and (5), derived in the form of eq.(12), we will have

$$T \leq 2k + 1. \tag{18}$$

The demonstration is direct for the functions (5). In fact, we have $T = S \leq 2k$ for the functions (3), from which follows the relation (18) for the functions (5).

In the case of the functions (4), the proof is conducted by induction from i to $i + 1$. Let us assume that the relation (18) is satisfied for these functions at $k = 1, 2, \dots i$ and let us demonstrate that it is thus also satisfied for $k = i + 1$.

For this purpose, let us return to eq.(6). Since the formulas (13) of Section 4 are given, we have $T \leq 2i + 3$ in the functions $\Phi'[i]$ and $\varphi'\{i\}$ which latter is a part of the function A'_{i+1}. In addition, in the part of A'_{i+1} which is obtained on dividing the expression (7) by ψ'_0, we obviously will have $T \leq 2i + 1$. The only item to be investigated are the parts of A'_{i+1} which are obtained from the expressions (9) by omitting there the factor ψ'_0. For these parts in accordance with our assumptions and in accordance with the data obtained for the functions (5), we will have

$$T \leq \sum_{k=1}^{i-1} (2k+1)(m'_k + n'_k + m''_k + n''_k) + \\ + 2m + 1 - \Sigma(m' + n' + m'' + n'') = 2i + 3.$$

Thus, we will definitely have $T \leq 2i + 3$ for the known terms of eq.(6), after having divided them by ψ'_0. It follows from this that the relation (18) is fulfilled for the functions (4) also for $k = i + 1$. However, this relation is fully satisfied for the functions (4) at $k = 1, 2$ since these functions then are polynomials of the third resp. fifth degree. Thus, theorem 3 is proved.

Below, we will need a more special proposition which we will demonstrate here:

<u>Theorem 4</u>: If k is even, no term with $j' + j'' = 2k - 1$ will exist in the function $\psi'_0\varphi'_k$ and no term with $j' + j'' = -(2k - 1)$ in the function $\varphi'_0\psi'_k$. /44

For the proof, we put $k = i + 1$, where i is odd.

It is sufficient to demonstrate that we have $j' + j'' \neq 2i + 1$ in the functions $\psi_0'\Phi'[i]$ and $\psi_0' A_{i+1}'$ which appear in eq.(6).

Moreover, in the function $\psi_0'\Phi'[i]$, we have $j' + j'' \neq 2i + 1$ since, because of i being odd, we have there $S = |j' + j''| \leq 2i$.

In the part (7) of the function $\psi_0' A_{i+1}'$, we also have $j' + j'' \neq 2i + 1$ since we always have there $S \leq 2i$ in accordance with theorem 2.

To find the terms where $j' + j'' = 2i + 1$ in the part (8) of the function $\psi_0' A_{i+1}'$, i.e., in the function $\psi_0'\Phi'\{i\}$ [see eq.(13) in Section 4], it is sufficient to consider, on the one hand, the terms of the functions $\{\psi_0'\varphi_i'\}$, ... $\{\varphi_0''\psi_i''\}$ where $j' + j'' = 2i$ and, on the other hand, the terms of $H^{(1)}$ that depend on ψ_0' and where $j' + j'' = 1$. (In fact, for the terms $H^{(1)}$ that depend on ψ_0', we never have $j' + j'' = 2$.) Moreover, according to theorem 2, it is only in the two functions $\{\varphi_0'\psi_i'\}$ and $\{\varphi_0''\psi_i''\}$ that we can ever have $j' + j'' = +2i$; in the expression of $\psi_0'\Phi'\{i\}$, these functions are accompanied by factors where $j' + j'' \neq$ $\neq +1$ in accordance with the Table given on p.20. Thus, in the part (8) of the function $\psi_0' A_{i+1}'$, we never have $j' + j'' = 2i + 1$.

Finally, let us pass to the parts (9) or (16) of the function $\psi_0' A_{i+1}'$. We know (p.32) that we always have $S \leq 2i + 3 - m$. Therefore, it is sufficient to consider the two cases in which $m = 2$ or 1.

In the case in which $m = 2$, it is sufficient to consider the terms of $H^{(2)}$ that depend on ψ_0' and where we have $j' + j'' = +3$. However, such terms cannot exist in virtue of the conditions (4) of Section 4 (for $m = 2$).

In the case in which $m = 1$, it is sufficient to consider the terms of $H^{(1)}$ that depend on ψ_0' and where we have $j' + j'' = 2$ or 1. However, in $H^{(1)}$ there is no term depending on ψ_0' where $j' + j'' = +2$. In addition, the terms of $H^{(1)}$ that depend on ψ_0' and where $j' + j'' = +1$, are as follows:

$$H^{1,3,0}_{1,0}\varphi_0'^2\psi_0' + H^{1,1,2}_{-1,2}\psi_0'\varphi_0''^2.$$

Using this expression of $H^{(1)}$, no functions $\varphi_0'\psi_k'$ and $\varphi_0''\psi_k''$ will exist in the /45
expression (16). In addition, in the functions $\psi_0'\varphi_k'$ and $\psi_0''\varphi_k''$ it is sufficient to consider the terms where $j' + j'' = +2k$. Moreover, according to theorem 2, such terms will exist only if k is even. However, in accordance with eq.(10) for $m = 1$, it is impossible that all k can be even since i is odd. Thus, in the parts (9) of the function $\psi_0' A_{i+1}'$, we will never have $j' + j'' = 2i + 1$.

We have demonstrated that $j' + j'' \neq 2i + 1$ in the wholly known terms of eq.(6), provided that i is odd. It follows from this that $j' + j'' \neq 2i + 1$ exists also in the function $\{\psi_0'\varphi_{i+1}'\}$ if i is odd. For reasons of symmetry, we will also have $j' + j'' \neq -(2i + 1)$ in the function $\{\varphi_0'\psi_{i+1}'\}$ if i is odd. Thus, the postulated theorem is proved.

Section 6.

Let us now substitute, in the expansions (8) of Section 4, the quantities φ_0', ψ_0', φ_0'', ψ_0'' by their expressions in accordance with the formulas (6) of Section 4. Then, φ' and φ'' take the form of the exponential series $e^{\sqrt{-1}(j'w'+j''w'')}$ with real coefficients. The functions ψ' and ψ'' are imaginaries conjugate with φ' and φ''.

In view of this and in accordance with formulas (24) of Section 3, the quantities ξ' and ξ'' become cosine series expanded in multiples of the arguments w' and w'' and in powers of μ. The quantities η' and η'' will be represented by sine series formed with the same coefficients that appear in the expansions for ξ' and ξ''.

Let us now introduce instead of the parameters ρ_0' and ρ_0'', two new parameters ϵ' and ϵ'', by setting

$$\epsilon' = \sqrt{\mu}\,\rho_0', \quad \epsilon'' = \sqrt{\mu}\,\rho_0''. \tag{1}$$

where ϵ' denotes the modulus of eccentricity and ϵ'' the modulus of inclination.

The quantities ρ_0' and ρ_0'' are comparable to unity. We will consider the small quantities ϵ', ϵ'', e', and $\sqrt{\mu}$ as being of the order of magnitude of one.

Let us set, in addition

$$\delta = \mu(\nu_1' + \nu_1''). \tag{2}$$

The quantity δ is homogeneous and linear with respect to ϵ'^2, ϵ''^2, e'^2, and μ /46 where the coefficients depend only on the parameter x_1^*. Thus, δ is a quantity of the order of magnitude of two.

Now, since the formulas (17) of Section 3 are given, the general solution of eqs.(1) of Section 3 takes the form

$$\begin{aligned}
\xi_1^* &= \bar{\xi} + \epsilon' \cos w' + \sum_{k=1}^{\infty} A_{j',j''}^{(2k+1)} \cos(j'w' + j''w''),\\
\eta_1^* &= \qquad \epsilon' \sin w' + \sum_{k=1}^{\infty} A_{j',j''}^{(2k+1)} \sin(j'w' + j''w''),\\
\xi_2^* &= \qquad \epsilon'' \cos w'' + \sum_{k=1}^{\infty} B_{j',j''}^{(2k+1)} \cos(j'w' + j''w''),\\
\eta_2^* &= \qquad \epsilon'' \sin w'' + \sum_{k=1}^{\infty} B_{j',j''}^{(2k+1)} \sin(j'w' + j''w'').
\end{aligned} \tag{3}$$

The quantity $\bar{\xi}:e'$, defined by eq.(16) of Section 3, can be expanded in powers of e'^2 and μ.

The coefficients $A^{(2k+1)}_{j',j''}$ and $B^{(2k+1)}_{j',j''}$ are of the order of magnitude of 2k + + 1. They are rational and homogeneous functions with respect to the quantities ϵ', ϵ'', e', $\sqrt{\mu}$. Only even powers of $\sqrt{\mu}$ are encountered. The denominator is a power δ^s of δ where the superscript s is ≥ 0 and $\leq 2k + 1 - 3$ if $j' - j'' = \pm 1$ but is $\leq 2k + 1 - 5$ if $j' - j'' \neq \pm 1$. Here, the plus sign refers to the coefficients A and the minus sign to the coefficients B. In view of theorem 3 on p.34, we have the inequality

$$|j'| + |j''| \leq 2k + 1.$$

In addition, j'' is even in ξ_1^* and η_1^* but odd in ξ_2^* and η_2^*. The numerator of $A^{(2k+1)}_{j',j''}$ and $B^{(2k+1)}_{j',j''}$ obviously contains the factor $\epsilon'^{|j'|} \cdot \epsilon''^{|j''|}$ if $j' + j''$ is odd and the factor $e'\epsilon'^{|j'|} \cdot \epsilon''^{|j''|}$ if $j' + j''$ is even. The other factor in this numerator is a polynomial homogeneous in ϵ'^2, ϵ''^2, e'^2, and μ.

Let us note that we have /47

$$A^{(2k+1)}_{+1,0} \equiv 0, \quad B^{(2k+1)}_{0,+1} \equiv 0, \qquad (k = 1, 2, \cdots \infty)$$

since we always have assumed

$$[[\psi'_0 \varphi'_k]] = [[\varphi'_0 \psi'_k]] = [[\psi''_0 \varphi''_k]] = [[\varphi''_0 \psi''_k]] = 0$$

during the integrations in Section 4. In addition, the coefficients

$$A^{(2k+1)}_{0,0} \qquad (k = 1, 2, \cdots \infty)$$

cancel out at ϵ' and ϵ'' since the special solution obtained by setting $\epsilon' = \epsilon'' = 0$ must coincide with the solution $\xi_1^* = \xi$, $\eta_1^* = 0$, $\xi_2^* = 0$, $\eta_2^* = 0$ investigated on p.14.

We have seen above that φ'_1, ψ'_1, φ''_1, ψ''_1 are polynomials and that the divisor $\nu'_1 + \nu''_1$ appears first in the functions $\frac{[\psi'_0\varphi'_1]}{\psi'_0}, \frac{[\varphi'_0\psi'_1]}{\varphi'_0}, \frac{[\psi''_0\varphi''_1]}{\psi''_0}, \frac{[\varphi''_0\psi''_1]}{\varphi''_0}$. Thus, the fractional inequalities of the variables ξ_k^*, η_k^* are at least of the order of magnitude of five. The fractional coefficients of the order of magnitude of five are $A^{(5)}_{4,2}$, $A^{(5)}_{-1,-2}$ and $B^{(5)}_{4,3}$, $B^{(5)}_{-2,-1}$.

Let us now pass to an integration of eqs.(2) of Section 3. We have already mentioned that the second of these equations will simply yield

$$y_2^* = t. \tag{4}$$

The first equation is written as

$$\frac{dy_1^*}{dt} = -\frac{dF^*}{dx_1^*} = x_1^{*-3} - \frac{d(F^* - F_0^*)}{dx_1^*}. \tag{5}$$

As derivative of the function $F^* - F_0^*$, we have an expansion obtained from the series (20) in Section 3 after differentiation with respect to x_1^*. Let us here introduce, by means of eqs.(21) and (23) of Section 3, first the variables φ_k^*, ψ_k^* and then the variables φ', ψ', φ'', ψ''. Next, in analogy with eqs.(19), (27), and (28) of Section 3, let us put /48

$$\frac{1}{\mu^2}\frac{d}{dx_1^*}(F^* - F_0^* - \mu F^{1,0,0,0}_{0,0,0,0}) = G = \sum_{m=0}^{\infty} \mu^m G^{(m)}, \tag{6}$$

$$G^{(m)} = \sum G^{m,\alpha'+\beta',\alpha''+\beta''}_{\alpha'-\beta',\alpha''-\beta''} \varphi'^{\alpha'} \psi'^{\beta'} \varphi''^{\alpha''} \psi''^{\beta''}. \tag{7}$$

The coefficients $G^{m,m',m''}_{j',j''}$ are polynomials in e_0. They can be derived by means of the formula

$$G^{m,m',m''}_{j',j''} = \sum \binom{\alpha_1}{\alpha'} \binom{\beta_1}{\beta'} \frac{dF^{i,m,m_1,m''}_{0,0,j',j''}}{dx_1^*} e_0^{i+m_1-m'} p_s^{(m_1-m')}, \tag{8}$$

in complete analogy with eq.(29) of Section 3. The relations (30) of Section 3 are also valid for the expression (8). The nonnegative whole numbers α', β', α'', β'' take only the values that satisfy the conditions

$$\begin{gathered} \alpha'' + \beta'' = \text{even} \\ 2m + 2 = |\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' + 2k \quad (k \geq 0) \end{gathered} \tag{9}$$

analogous to the conditions (31) of Section 3. Obviously, the function G does not change on permuting φ' and ψ' as well as φ'' and ψ''.

In the function G, let us now introduce for φ', ψ', φ'', ψ'' their expansions (8) of Section 4 and let us arrange this series in powers of μ. This will yield

$$G = \sum_{m=0}^{\infty} \mu^m G^{(m)} = \sum_{i=0}^{\infty} \mu^i g^{(i)}. \tag{10}$$

In view of this, eq.(5) can be written as

$$\frac{dy_1^*}{dt} = x_1^{*-3} - \mu \frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1^*} - \mu^2 \sum_{i=0}^{\infty} \mu^i g^{(i)}. \tag{11}$$

We will now investigate the function $g^{(1)}$. It is obvious that $g^{(1)}$ is even with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0'' and that $g^{(1)}$ does not change on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''. It is also obvious that the function $g^{(1)}$ is /49 composed of two terms of the form

$$\varphi_1'^{m_1'}\psi_1'^{n_1'}\varphi_1''^{m_1''}\psi_1''^{n_1''}\ldots\varphi_i'^{m_i'}\psi_i'^{n_i'}\varphi_i''^{m_i''}\psi_i''^{n_i''}\frac{d^{\Sigma(m'+n'+m''+n'')}G^{(m)}}{(d\varphi_0')^{\Sigma m'}(d\psi_0')^{\Sigma n'}(d\varphi_0'')^{\Sigma m''}(d\psi_0'')^{\Sigma n''}} \tag{12}$$

multiplied by certain factorials. The nonnegative whole numbers m_k', n_k', m_k'', n_k'' satisfy the condition

$$\sum_{k=1}^{i} k\,(m_k' + n_k' + m_k'' + n_k'') + m = i. \tag{13}$$

Let us first define the degree of the rational function $g^{(i)}$. Since theorem 1 of Section 5 is given, the degree of the general term (12) can be at most

$$\sum_{k=1}^{i}(2k+1)(m_k' + n_k' + m_k'' + n_k'') + \\ + 2m + 2 - \Sigma(m' + n' + m'' + n'') = 2i + 2.$$

Thus, the rational function $g^{(i)}$ at most is of the degree $2i + 2$ with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0''.

Let us next investigate at what power the quantity $\nu_1' + \nu_1''$ enters the denominators of the various terms of the rational function $g^{(i)}$, put into the form (12) of Section 5. We state that this power is at most $2i - 2$ for the terms where $j' - j'' = \pm 1$ and at most $2i - 4$ for all other terms.

In fact, in accordance with theorem 1 (on p.29), the power of $\nu_1' + \nu_1''$ in the denominator of the expression (12) can be at most

$$\sum_{k=1}^{i}(2k-2)(m_k' + n_k' + m_k'' + n_k'') = \\ = 2i - 2m - 2\Sigma(m' + n' + m'' + n'').$$

Several cases must be differentiated, in accordance with the value of m and with the order Σ of the derivative in eq.(12).

If $\Sigma = 0$, the expression (12) reduces to $G^{(i)}$ and will have no denominator at all. /50

If $\Sigma > 1$, the power of $\nu_1' + \nu_1''$ will be at most $2i - 4$.

If $\Sigma = 1$ and $m > 0$, the power of $\nu_1' + \nu_1''$ will be at most $2i - 4$.

If $\Sigma = 1$ and $m = 0$, the expression (12) is reduced to

$$\varphi_i'\frac{dG^{(0)}}{d\varphi_0'} + \psi_i'\frac{dG^{(0)}}{d\psi_0'} + \varphi_i''\frac{dG^{(0)}}{d\varphi_0''} + \psi_i''\frac{dG^{(0)}}{d\psi_0''}. \tag{14}$$

Moreover, in accordance with eq.(7) and the conditions (9), the function $G^{(0)}$

has the expression

$$G^{(0)} = G^{0,0,0}_{0,0} + G^{0,1,0}_{1,0}(\varphi' + \psi') + G^{0,2,0}_{0,0}\varphi'\psi' + G^{0,0,2}_{0,0}\varphi''\psi''.$$

In addition, the general formula (8) as well as eq.(38) in Section 3 showed that

$$G^{0,0,2}_{0,0} = \frac{dF^{1,0,0,2}_{0,0,0,0}}{dx_1^*} = -\frac{dF^{1,0,2,0}_{0,0,0,0}}{dx_1^*} = -G^{0,2,0}_{0,0}.$$

Thus, the expression (14) becomes

$$G^{0,1,0}_{1,0}(\varphi'_i + \psi'_i) + G^{0,2,0}_{0,0}(\psi'_0\varphi'_i + \varphi'_0\psi'_i - \psi''_0\varphi''_i - \varphi''_0\psi''_i).$$

This is the part of the function $g^{(i)}$ which, in the denominator, could include the quantity $\nu'_1 + \nu''_1$ raised to a power $> 2i - 4$. However, we know that, in the denominators of the functions $\{\psi'_0\varphi'_i\}$, $\{\varphi'_0\psi'_i\}$, $\{\psi''_0\varphi''_i\}$, $\{\varphi_0\psi_i\}$, the quantity $\nu'_1 + \nu''_1$ enters at most at the power $2i - 4$. Thus, the part of the function $g^{(i)}$ which, in the denominator, could include the quantity $\nu'_1 + \nu''_1$, raised to a power $> 2i - 4$, will simply be

$$G^{0,1,0}_{1,0}\left(\frac{[\psi'_0\varphi'_i]}{\psi'_0} + \frac{[\varphi'_0\psi'_i]}{\varphi'_0}\right) + \\ + G^{0,2,0}_{0,0}([\psi'_0\varphi'_i + \varphi'_0\psi'_i] - [\psi''_0\varphi''_i + \varphi''_0\psi''_i]). \tag{15}$$

Now, we state that, in the denominator the function /51

$$[\psi'_0\varphi'_i + \varphi'_0\psi'_i] - [\psi''_0\varphi''_i + \varphi''_0\psi''_i], \tag{16}$$

the quantity $\nu'_1 + \nu''_1$ enters at most at the power $2i - 4$. In fact, let us start from the first integral

$$H = \sum_{i=0}^{\infty} \mu^i H^{(i)} = \text{const.}$$

of the system (1) in Section 4. The functions $H^{(n)}$ are defined by the formula (3) of Section 4. On substituting there the quantities φ', ψ', φ'', ψ'' by their expansions (8) from Section 4 and by arranging the series in powers of μ, the integral in question is written as

$$\sum_{i=0}^{\infty} \mu^i H^{(i)} \equiv \sum_{i=0}^{\infty} \mu^i h^{(i)} = \text{const.} \tag{17}$$

We can argue on the function $h^{(i)}$ as we had done above for $g^{(i)}$. It is then sufficient to replace everywhere $G^{(n)}$ by $H^{(n)}$. Thus, it will become obvious that the terms of $h^{(i)}$, in which the quantity $\nu'_1 + \nu''_1$ may enter raised to a power $> 2i - 4$, are contained in the expression

$$H^{0,2,0}_{0,0}([\psi'_0\varphi'_i+\varphi'_0\psi'_i]-[\psi''_0\varphi''_i+\varphi''_0\psi''_i]).$$

Evidently, because of the investigated first integral, these terms in which the quantity $\nu'_1 + \nu''_1$ enter the denominator raised to a power $> 2i - 4$ must cancel out and vanish (since they are not constant). Thus, it is obvious that, in the denominator of the expression (16), the quantity $\nu'_1 + \nu''_1$ enters at most at the power $2i - 4$.

Let us now return to the expression (15). We have demonstrated that the terms of $g^{(i)}$ which, in their denominator, could include the quantity $\nu'_1 + \nu''_1$ raised to a power $> 2i - 4$ must necessarily be present in the expression

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$$G^{0,1,0}_{1,0}\left(\frac{[\psi'_0\varphi'_i]}{\psi'_0}+\frac{[\varphi'_0\psi'_i]}{\varphi'_0}\right). \tag{18}$$

In the monomials $\mathfrak{M}$ of this function, given the form (12) of Section 5, we have $j' - j'' = \pm 1$. In the denominator, the quantity $\nu'_1 + \nu''_1$ may enter raised to at most a power of $2i - 2$, in accordance with theorem 1 on p.29.

Finally, let us define an upper limit for the numbers $T = |j'| + |j''| = |\alpha' - \beta'| + |\alpha'' - \beta''|$ in the function $g^{(i)}$, put into the form of eq.(12) of Section 5. We now state that we still have $T \leq 2i + 2$ and that, in the part $[g^{(i)}]$, we even have $T \leq 2i - 2$.

In fact, in view of theorem 3 on p.34, we will have the following expression in the part (12) of the function $g^{(i)}$:

$$T \leq \sum_{k=1}^{i}(2k+1)(m'_k+n'_k+m''_k+n''_k) + \\ + 2m + 2 - \Sigma(m'+n'+m''+n'') = 2i+2.$$

Thus, in the function $g^{(i)}$ thrown into the form (12) of Section 5, we still have $T \leq 2i + 2$.

To go further, we put the expression (12) in the following form:

$$(\psi'_0\varphi'_1)^{m'_1}(\varphi'_0\psi'_1)^{n'_1}(\psi''_0\varphi''_1)^{m''_1}(\varphi''_0\psi''_1)^{n''_1}\ldots \\ (\psi'_0\varphi'_i)^{m'_i}(\varphi'_0\psi'_i)^{n'_i}(\psi''_0\varphi''_i)^{m''_i}(\varphi''_0\psi''_i)^{n''_i} \\ \frac{d^{\Sigma(m'+n'+m''+n'')}G^{(m)}}{(\psi'_0 d\varphi'_0)^{\Sigma m'}(\varphi'_0 d\psi'_0)^{\Sigma n'}(\psi''_0 d\varphi''_0)^{\Sigma m''}(\varphi''_0 d\psi''_0)^{\Sigma n''}}. \tag{19}$$

For each of the functions $\psi'_0\varphi'_k$, $\varphi'_0\psi'_k$, $\psi''_0\varphi''_k$, $\varphi''_0\psi''_k$, placed into the form of eq.(12) of Section 5, we always have $S = |j' + j''| \leq 2k$, in accordance with theorem 2 postulated on p.32. In addition, in the function on the second line of eq.(19), we have $S \leq m + 1$ as in the function $G^{(m)}$, because of the second condition (9). In view of this, we thus have the following expression in the function (19), given the form of eq.(12) of Section 5: /53

$$S \leq \sum_{k=1}^{i} 2k(m'_k + n'_k + m''_k + n''_k) + m + 1 = 2i + 1 - m \leq 2i + 1.$$

In the part $[g^{(i)}]$ of the function $g^{(i)}$, we have $j' = j'' =$ even, so that S and T coincide and are divisible by 4. Thus, we definitely have $T \leq 2i - 2$ in the function $[g^{(i)}]$ if i is odd.

The case in which i is an even number remains to be investigated.

Let us first, generally, derive the parts (19) of the function $g^{(i)}$ in which we could have $S \geq 2i$. Evidently, it is sufficient to consider the parts (19) where $m = 1$ or $m = 0$. If $i > 1$, which we assume here, it is sufficient to consider the parts (19) where $\Sigma(m' + n' + m'' + n'') \geq 1$. In addition, in $G^{(1)}$ it is sufficient to consider only the terms where $S = 2$, i.e., the terms

$$G_{2,0}^{1,2,0}(\varphi'^2 + \psi'^2) + G_{0,2}^{1,0,2}(\varphi''^2 + \psi''^2).$$

(see the Table on p.20). Thus, the terms of $g^{(i)}$ in which we could have $S \geq 2i$ are necessarily located in the part

$$\begin{aligned}
&G_{1,0}^{0,1,0}\left(\frac{\psi'_0\varphi'_i}{\psi'_0} + \frac{\varphi'_0\psi'_i}{\varphi'_0}\right) \\
&+ G_{0,0}^{0,2,0}\sum_{a+b=i}\left(\frac{(\psi'_0\varphi'_a)(\varphi'_0\psi'_b)}{\varphi'_0\psi'_0} - \frac{(\psi''_0\varphi''_a)(\varphi''_0\psi''_b)}{\varphi''_0\psi''_0}\right) \\
&+ G_{2,0}^{1,2,0}\sum_{c+d=i-1}\left(\frac{(\psi'_0\varphi'_c)(\psi'_0\varphi'_d)}{\psi'^2_0} + \frac{(\varphi'_0\psi'_c)(\varphi'_0\psi'_d)}{\varphi'^2_0}\right) \\
&+ G_{0,2}^{1,0,2}\sum_{c+d=i-1}\left(\frac{(\psi''_0\varphi''_c)(\psi''_0\varphi''_d)}{\psi''^2_0} + \frac{(\varphi''_0\psi''_c)(\varphi''_0\psi''_d)}{\varphi''^2_0}\right).
\end{aligned} \tag{20}$$

Now, we make the statement that we have $S \neq 2i$ in the function $g^{(i)}$, provided that the number i is even.

In fact, we then have $S \neq 2i$ in the first line of eq.(20) since $j' + j'' \neq \pm 2i - 1$ in the function $\psi'_0\varphi'_i$ and since $j' + j'' \neq \pm 2i + 1$ in the function $\varphi''_0\psi''_i$ (see theorem 4 on p.34).

Let us then investigate the general term corresponding to the indices c, d in the third line of eq.(20). In order to have $j' + j'' = +2i$, we must pre- /54 serve, in $\psi'_0\varphi'_c$, the terms where $j' + j'' = +2c$ and, in $\psi'_0\varphi'_d$, the terms where $j' + j'' = +2d$. Similarly, to have there $j' + j'' = -2i$, it is necessary to retain, in $\varphi'_0\psi'_c$, the terms where $j' + j'' = -2c$ and, in $\varphi'_0\psi'_d$, the terms where $j' + j'' = -2d$. Moreover, one of the numbers c, d is odd since their sum in odd ($= i - 1$). Let k be the odd number here. According to theorem 2 (p.32), we have $j' + j'' \neq +2k$ in $\psi'_0\varphi'_k$ and $j' + j'' \neq -2k$ in $\varphi'_0\psi'_k$. Thus, we have $S \neq 2i$ in the third line of eq.(20).

It can be demonstrated in the same manner that $S \neq 2i$ in the fourth line.

The second line in eq.(20) remains to be investigated. We state that also there we have $S \neq 2i$, provided that i is even. In fact, we can argue on the function $h^{(i)}$ introduced by eq.(17), as we had argued above on the function $g^{(i)}$. It will be found that all the terms of $h^{(i)}$, where $S = 2i$, must be located in the following part of $h^{(i)}$:

$$H_{0,0}^{0,2,0} \sum_{a+b=i} \left(\frac{(\psi'_0 \varphi'_a)(\varphi'_0 \psi'_b)}{\varphi'_0 \psi'_0} - \frac{(\psi''_0 \varphi''_a)(\varphi''_0 \psi''_b)}{\varphi''_0 \psi''_0} \right). \tag{21}$$

Moreover, $h^{(1)}$ is a constant. Consequently, we have $S \neq 2i$ in the expression (21) and also in the second line of eq.(20).

We now have demonstrated that $S \neq 2i$ in eq.(20) and thus also in the function $g^{(i)}$, provided that i is even.

It follows from this that $T \neq 2i$ in the function $[g^{(i)}]$. We have demonstrated above that $T \leq 2i$ in this function. Therefore, we will still have $T \leq$ $\leq 2i - 2$ in the function $[g^{(i)}]$.

Let us now introduce, in the function $g^{(i-1)}$, the parameters ρ_0' and ρ_0'' as well as the arguments w' and w'', defined on p.20. Then, we can write the function $g^{(i-1)}$ in the form

$$g^{(i-1)} = \sum_{j',j''} G^{(2i)}_{j',j''} \cos (j'w' + j''w''), \tag{22}$$

with the condition $j' \geq j''$. According to what we demonstrated with respect to the function $g^{(i)}$, it is obvious that $G^{(2i)}_{j',j''}$ is rational and even, having the degree 2i with respect to e_0, ρ_0', ρ_0''. The denominator of $G^{(2i)}_{j',j''}$ is of the form $(\nu_1' + \nu_1'')^s$. We have $s \geq 0$; $s \leq 2i - 4$, if $j' - j'' = 1$; $s \leq 2i - 6$ if $j' - j'' \neq$ $\neq 1$. Finally, we still have $|j'| + |j''| \leq 2i$ and even $|j'| + |j''| \leq 2i - 4$ if $j' = j''$. /55

After all this preparatory work, we can pass to the integration of eq.(11). Let us denote the constant term of the second member by n. Since

$$w' = \mu \nu' t + \gamma', \quad w'' = \mu \nu'' t + \gamma'',$$

we will obtain, after integration,

$$y_1^* = nt + c - \sum' \frac{\mu^i G^{(2i)}_{j',j''}}{j'\nu' + j''\nu''} \sin (j'w' + j''w''), \tag{23}$$

where c is an arbitrary constant. In the sum Σ', all terms where $j' = j'' = 0$ must be excluded.

It is now a question of expanding the quantity $(j'\nu' + j''\nu'')^{-1}$ in order of magnitude of the individual terms. In eq.(8) of Section 4, we gave the expansions for ν' and ν''. We know that

$$\nu'_0 + \nu''_0 = 0;$$

and we also know that ν'_s and ν''_s are rational in e_0^2, $\rho_0'^2$, $\rho_0''^2$ of the degree 2s with respect to e_0, ρ'_0, ρ''_0 since the denominator is a power of $\nu'_1 + \nu''_1$ whose exponent is $\leq 2s - 6$.

Let us now fix j' and j'' by first assuming that $j' \neq j''$. By arranging the series in powers of μ, we can then put

$$\sum_{2i \geq |j'|+|j''|} \frac{\mu^i G^{(2i)}_{j',j''}}{j'\nu' + j''\nu''} = \sum_{2k \geq |j'|+|j''|} \mu^k \Gamma^{(2k)}_{j',j''}. \tag{24}$$

The quantity $\Gamma^{(2k)}_{j',j''}$ is rational and has the degree 2k with respect to e_0, ρ'_0, ρ''_0. Its denominator is a power of $\nu'_1 + \nu''_1$ since the exponent is $\leq 2k - 4$ if $j' - j'' = 1$ and $\leq 2k - 6$ if $j' - j'' \neq 1$. /56

Let us next assume that $j' = j'' = j$. In this case, we have

$$(j'\nu' + j''\nu'')^{-1} = \frac{1}{j\mu(\nu'_1 + \nu''_1)}\left(1 + \mu\frac{\nu'_2 + \nu''_2}{\nu'_1 + \nu''_1} + \mu^2\frac{\nu'_3 + \nu''_3}{\nu'_1 + \nu''_1} + \cdots\right)^{-1}.$$

Let us then put

$$\sum_{i=|j|+2}^{\infty} \frac{\mu^i G^{(2i)}_{j,j}}{j(\nu' + \nu'')} = \sum_{k=|j|+1}^{\infty} \mu^k \Gamma^{(2k)}_{j,j}. \tag{25}$$

The quantity $\Gamma^{(2k)}_{j;j''}$ is rational and has the degree 2k with respect to e, ρ'_0, ρ''_0. Its denominator is a power of $\nu'_1 + \nu''_1$ whose exponent is $\leq 2k - 3$.

Finally, let us introduce the quantities ϵ', ϵ'' and δ defined by eqs.(1) and (2). In addition, let us put

$$-\mu^k \Gamma^{(2k)}_{j',j''} = C^{(2k)}_{j',j''}.$$

Then, in view of eqs.(23), (24), and (25), the solution of the first equation in the system (2) of Section 3 takes the form

$$y_1^* = nt + c + \sum_{k=1}^{\infty} C^{(2k)}_{j',j''} \sin(j'w' + j''w''). \tag{26}$$

This is a formula analogous to the formulas (3). The coefficients $C^{(2k)}_{j',j''}$ are of the order of magnitude of 2k; they are rational and homogeneous with respect to the quantities ϵ', ϵ'', e'_s, and $\sqrt{\mu}$. Only even powers of $\sqrt{\mu}$ are encountered. The denominator is a power δ^s of δ since the exponent s is ≥ 0 and

$$\begin{aligned} &\leq 2k-3, && \text{if } j'-j''=0; \\ &\leq 2k-4, && \text{if } j'-j''=1; \\ &\leq 2k-6, && \text{if } j'-j''>1. \end{aligned}$$

The integers j' and j'' are limited by the conditions /57

$$\begin{aligned} &j'-j'' \geq 0; \qquad j''=\text{even}; \\ &0<j'+j''\leq 2k-2, && \text{if } j'-j''=0; \\ &|j'|+|j''|\leq 2k, && \text{if } j'-j''>0. \end{aligned}$$

The numerator of $C^{(2k)}_{j',j''}$ contains the factor $\epsilon'^{|j'|}\ \epsilon''^{|j''|}$ if $j' - j''$ is even and the factor $e'\epsilon'^{|j'|}\epsilon''^{|j''|}$ if $j' - j''$ is odd. The other factor of this numerator is a polynomial homogeneous in ϵ'^2, ϵ''^2, e'^2, and μ.

Obviously, the coefficients $C^{(2)}_{j',j''}$ and $C^{(4)}_{j',j''}$ are polynomials. The fractional inequalities of the longitude are at least of the order of magnitude of six. The fractional coefficients of the order of magnitude of six are $C^{(6)}_{3,2}$, $C^{(6)}_{-1,-2}$, and $C^{(6)}_{2,2}$. The first two coefficients originate in the function $g^{(2)}$ [the part (18) for $i = 2$]. The third coefficient has its origin in $[g^{(3)}]$.

The quantity n is known as the mean absolute motion of the planet and can be expanded in the form of

$$n = n^{(0)} + n^{(2)} + n^{(4)} + \cdots$$

In accordance with eq.(11), we specifically have

$$n^{(0)} = x_1^{*-3}, \quad n^{(2)} = -\mu \frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1^*}.$$

The quantity $n^{(2k)}$ is of the order of magnitude 2k; it is rational and homogeneous with respect to ϵ'^2, ϵ''^2, e'^2, and μ. The denominator of $n^{(2k)}$ is a power δ^s where the exponent s satisfies the conditions $0 \leq s \leq 2k - 8$. Thus, $n^{(4)}$, $n^{(6)}$, $n^{(8)}$ are polynomials. In addition, the $n^{(2k)}$ ($k = 1, 2, \ldots \infty$) obviously contain μ as the factor.

After having integrated eqs.(1) and (2) of Section 3, we should briefly discuss the system (3) of Section 2. This system yields the general solution (except for the auxiliary variable x_2 which need not be known) by introducing, in eqs.(14) of Section 3, the expressions (26) and (3) of the variables y_1^*, ξ_1^*, η_1^*, ξ_2^*, η_2^*. Thus, the unknowns x_1, $y_1 - (nt + c)$, ξ_1, η_1, ξ_2, η_2 are expanded /58 in trigonometric series, arranged in multiples of the four arguments linear with respect to time:

$$nt + c, \ \ t, \ \ w' = \mu\nu' t + \gamma', \ \ w'' = \mu\nu'' t + \gamma''.$$

The coefficients of these trigonometric series are series of functions rational and homogeneous in ϵ', ϵ'', e', and $\sqrt{\mu}$, arranged in order of magnitude of the terms. Here, only even powers of $\sqrt{\mu}$ are encountered. The denominators of the rational functions are powers of the quantity δ. However, it obviously is not necessary to enter into more details of this solution whose general nature is sufficiently well known from the above discussion.

One of the essential points of the theory of ordinary minor planets is the appearance of the quantity δ in the denominators. This quantity will always have at least an order of magnitude of two. In reality, the series given in Sections 4 and 5 are expanded in powers of quantities comparable to $\mu^3\delta^{-2}$. Thus, the discussed integration method basically assumes that δ is not comparable to $\mu^{3/2}$ or still smaller. In Part II of this research, we will investigate the exceptional cases in which δ becomes too small, so that the terms of the series used do not converge sufficiently rapidly.

RESEARCH ON THE MOTION OF MINOR PLANETS

PART II

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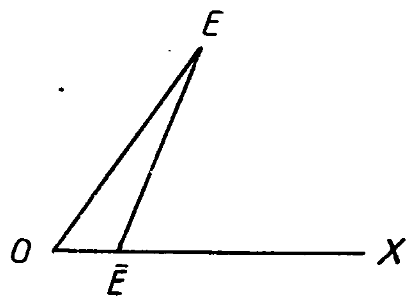
In this Part II of the research on the motion of minor planets, we will discuss the theory of ordinary planets in a general manner which is applicable not only to the case of regular planets treated in Part I of this report but also to the case of planets which we there called "singular". To accurately define the difference between regular planets and singular planets, we will give in this Introduction some information of a geometric nature. These new concepts will make it possible to discuss here, in relatively few words, the main results obtained in this second part of the research.

By eliminating, from the theory of ordinary planets, the moduli of eccentricity and inclination, we will obtain a particular solution containing only two arbitrary constants. The canonical elements in this solution are periodic functions of two arguments linear with respect to time and having the mean absolute motion of the two planets as velocities. The two arguments of slow motion have vanished together with the moduli. In this particular solution, the inclination is zero; the perihelion executes small oscillations about the perihelion of the orbit of Jupiter; the eccentricity is small and nearly constant. /2



Let us designate by "normal eccentricity" the mean value of the eccentricity in this particular solution. The normal eccentricity, which we denote here by $\bar{e}$, depends only on the mean absolute motion of the minor planet. The ratio of $\bar{e}$ to the eccentricity e' of the orbit of Jupiter can be expanded in powers of e'^2 and of μ (mass of Jupiter).

After this, we will consider the orbit of an arbitrary ordinary minor

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planet. To represent the motion of its perihelion, we will define three vectors $O\bar{E}$, OE, and $\bar{E}E$ in any fixed plane. The vector $O\bar{E}$ is fixed and its magnitude is equal to $\bar{e}$. The vector OE is variable and, for magnitude, has the osculatory eccentricity of the investigated general orbit. In addition, the angle XOE is to represent the longitude of the perihelion of this orbit, counted from the perihelion of Jupiter. Finally, the vector $\bar{E}E$ will be the geometric difference of the two vectors OE and $O\bar{E}$.

The vector $\bar{E}E$ will be designated as the "eccentric vector", while the angle $X\bar{E}E$ will be called the "longitude of the eccentric vector".

Because of the perturbations of Jupiter, the point E describes a certain curve in the plane under consideration. This curve is almost a circle about the point $\bar{E}$ as center. The nonperiodic component of the longitude of the eccentric vector defines a linear argument with respect to time, which we have denoted by

$$-w' = -(\mu\nu' t + \gamma').$$

The velocity $-\mu\nu'$ always is a positive constant of the order of μ. 13

We will also consider the longitude of the node of the general orbit in question, with this longitude being counted in the plane of the orbit of Jupiter starting from its perihelion. The nonperiodic portion of the longitude of the node defines a second argument, linear with respect to time. We have denoted this by

$$-w'' = -(\mu\nu'' t + \gamma'').$$

The velocity $-\mu\nu''$ always is a negative quantity of the order of μ.

In the theory of ordinary minor planets, the velocity

$$\mu(\nu' + \nu'')$$

is always very small, at least of the order of μ^2. If the velocity actually is of the order of μ^2, we have to do with a regular planet. If, conversely, the velocity $\mu(\nu' + \nu'')$ is of the order of $\mu^{5/2}$ or even smaller, the planet will be singular. Singular planets exist for which the velocity $\mu(\nu' + \nu'')$ is as small as desired. There are other planets, completely of the general type, for which the quantity $\nu' + \nu''$ is identically zero. In this latter case, a libration between the longitude of the eccentric vector and the longitude of the node exists. In the case of libration, the two arguments w' and w'' are no longer independent; their sum is a multiple of $\frac{\pi}{2}$. To compensate, a new linear argument w is introduced which can be called the argument of libration. The velocity of this argument w is at least of the order of μ^3 but can also be infinitely small.

Let us now define the analytical form of the canonical elements in the theory of ordinary planets. Let t be the mean anomaly of Jupiter (whose motion is supposed to be of the Keplerian type with the eccentricity e'). Let, in addition, n be the mean absolute motion of the minor planet. The canonical elements are trigonometric series of the two arguments

$$t \text{ and } nt + c.$$

The coefficients of these series are slowly variable and can be expanded in /4
powers of certain quantities, comparable in magnitude to the eccentricities, the inclination, and the square root of the mass μ of Jupiter. We express this by stating that the coefficients in question can be expanded in powers of $\sqrt{\mu}$. What remains to be defined is the nature of the slowly variable coefficients C which, in these expansions, multiply the various powers of $\sqrt{\mu}$.

In the case of regular planets and for certain types of singular planets, the coefficients C are polynomials in cosine and sine of the two arguments w' and w''.

For other types of singular planets, specifically in the case of libration which we mentioned above, it is convenient to introduce elliptical functions snv, cnv, dnv and their integrals, in which case the argument v of these functions will be

$$\frac{2K}{\pi}(w' + w'')$$

provided that no libration is present, and

$$\frac{2K}{\pi}w$$

in the case of libration (K is the real half-period of the elliptical functions). Then, the coefficients C are polynomials with respect to these elliptical functions and their integrals, as well as with respect to the cosine and sine of the principal portion of the longitude of the node*. The expansions of the coefficients C, placed in this form, never fail. In some cases, they are preferable over the corresponding two-argument trigonometric series. In fact, the terms of these latter series converge too slowly when the velocity of the argument $w' + w''$ resp. w becomes too low, which happens whenever the modulus of the elliptical functions approaches unity.

In conformity with the program of our research, which has the specific /5
purpose of a qualitative investigation of the motion, we will not give all details of such expansions. We rather limit the discussion here to indicating the form of the series and the order of magnitude of their principal terms. In addition, we will show how these series depend on arbitrary parameters. This more or less constitutes the basic contents of the various Sections of this Part II of our report.

First, in Section 7 we will demonstrate the possibility of the existence

* This principal portion includes first the term $-w''$ which is linear with respect to time and then a certain periodic term with the period 2π with respect to the argument $w' + w''$ resp. w. This periodic term is comparable, in magnitude, to unity.

of singular planets by showing that the divisor δ (see Part I, p.36) may vanish for the eigenvalues of the arbitrary parameters.

In Section 8, the general method of Section 1 will yield the means for reducing the equations of secular variations to one degree of freedom.

Next, in Section 9 we will investigate, by geometric means, the nature of the general solution of this simple canonical system. Occasionally, the contours defined by the first integral of the investigated canonical system contain a double-point curve. This curve constitutes the limits of the libration domains mentioned above.

In Sections 10 - 12, we will demonstrate that it is still possible to form the solution of the investigated canonical system by means of series whose first terms converge quite rapidly. In the vicinity of the double-point contour curve, which forms the limit of the libration case, the use of elliptical functions becomes indispensable.

Ultimately, in Sections 13 - 15, we will continue the integrations to finally obtain general series giving the canonical elements of ordinary minor planets as functions of time and of six arbitrary parameters.

If the values of the arbitrary constants were randomly distributed, singular planets would occur rather rarely. In that case, their number would be only a few percent of the total number of asteroids. Planets characterized by libration between the longitude of the eccentric vector and the longitude of the node would be even less likely to occur. Nevertheless, it is well possible, because of the resistance of space, which probably has been much greater in the past, /6 that the values of the arbitrary constants have undergone changes and that they thus had tended finally to satisfy the conditions of singular planets. In fact, the libration centers correspond exactly to the minimum values of the total energy (denoted by $-F$) if we vary the value of the parameters which we denoted by h. It is well possible that a study of the distribution of the values of the arbitrary constants in the theory of motion of minor planets may lead toward the discovery of this mysterious resistance. However, before approaching this question it is necessary to investigate the various types of motion and to check how the solution depends on the arbitrary parameters.

Section 7.

To demonstrate that the minor planets, designated as "singular" planets, might actually exist, we will show that the quantity δ, or else the quantity $\nu_1' + \nu_1''$, may cancel out. For abbreviation, we will put

$$\begin{aligned} N' &= -(2H_{0,0}^{1,4,0} + H_{0,0}^{1,2,2}), \\ N'' &= -(2H_{0,0}^{1,0,4} + H_{0,0}^{1,2,2}), \\ N &= H_{0,0}^{1,2,0} + H_{0,0}^{1,0,2}. \end{aligned} \qquad (1)$$

Then, the quantity $\nu_1' + \nu_1''$, introduced in Section 4, can be written in the form of

$$\nu_1' + \nu_1'' = 2(N'\varrho_0'^2 + N''\varrho_0''^2 - N). \tag{2}$$

First, we will give the expressions of the coefficients $H_{0,0}^{1,n,n'}$ in accordance with eq.(29) of Section 3:

$$\begin{aligned} &H_{0,0}^{1,4,0} = F_{0,0,0,0}^{1,0,4,0}, \quad H_{0,0}^{1,2,2} = F_{0,0,0,0}^{1,0,2,2}, \quad H_{0,0}^{1,0,4} = F_{0,0,0,0}^{1,0,0,4}, \\ &H_{0,0}^{1,2,0} = F_{0,0,0,0}^{2,0,2,0} + e_0^2 (F_{0,0,0,0}^{1,2,2,0} + 4F_{0,0,1,0}^{1,1,3,0} p_0^{(1)} + 4F_{0,0,0,0}^{1,0,4,0} p_0^{(2)}), \\ &H_{0,0}^{1,0,2} = F_{0,0,0,0}^{2,0,0,2} + e_0^2 F_{0,0,0,0}^{1,2,0,2}. \end{aligned} \tag{3}$$

In view of eq.(26) of Section 3 and the first term in the expansion for the root of eq.(16) of Section 3, we also have /7

$$p_0^{(1)} = -F_{0,0,1,0}^{1,1,1,0} : F_{0,0,0,0}^{1,0,2,0}, \quad p_0^{(2)} = (p_0^{(1)})^2.$$

It is easy to express the coefficients $F_{0,0,0,0}^{1,0,2,0}$, $F_{0,0,0,0}^{1,0,0,2}$, $F_{0,0,0,0}^{1,0,4,0}$, $F_{0,0,0,0}^{1,0,2,2}$, $F_{0,0,0,0}^{1,0,0,4}$ by means of Laplace functions. To this end, we will retain, in the expansion of the perturbation function [eq.(5) of Section 2] the terms which are independent of e' and of the four arguments y_1, y_2, ω_1, ω_2. Below, we give these terms up to the fourth degree inclusive (excluding the constant term):

$$F_{0,0,0,0}^{1,0,2,0}\varrho_1^2 + F_{0,0,0,0}^{1,0,0,2}\varrho_2^2 + F_{0,0,0,0}^{1,0,4,0}\varrho_1^4 + F_{0,0,0,0}^{1,0,2,2}\varrho_1^2\varrho_2^2 + F_{0,0,0,0}^{1,0,0,4}\varrho_2^4 + \cdots.$$

Moreover, in the Leverrier theory, the following expansion is obtained for this part of the perturbation function

$$(2)^{(0)}\left(\frac{e}{2}\right)^2 + (11)^{(0)}\eta^2 + (4)^{(0)}\left(\frac{e}{2}\right)^4 + (12)^{(0)}\left(\frac{e}{2}\right)^2\eta^2 + (17)^{(0)}\eta^4 + \cdots$$

where the coefficients depend on the major semiaxes a and a'. For comparing the two series, it is necessary to put, in the latter,

$$\begin{gathered} a = x_1^2, \quad a' = 1, \\ \left(\frac{e}{2}\right)^2 = \frac{\varrho_1^2}{4x_1} - \left(\frac{\varrho_1^2}{4x_1}\right)^2, \\ \eta^2 = \frac{\varrho_2^2}{4x_1 - 2\varrho_1^2} = \frac{\varrho_2^2}{4x_1} + 2\frac{\varrho_1^2}{4x_1}\frac{\varrho_2^2}{4x_1} + \cdots \end{gathered}$$

(see the equations in Section 2). In this manner, a comparison of the two series will yield the following formulas

$$\begin{gathered} 4x_1 F_{0,0,0,0}^{1,0,2,0} = (2)^{(0)}, \\ 4x_1 F_{0,0,0,0}^{1,0,0,2} = (11)^{(0)}, \end{gathered}$$

$$16x_1^2 F_{0,0,0,0}^{1,0,4,0} = -(2)^{(0)} + (4)^{(0)},$$

$$16x_1^2 F_{0,0,0,0}^{1,0,2,2} = 2(11)^{(0)} + (12)^{(0)},$$

$$16x_1^2 F_{0,0,0,0}^{1,0,0,4} = (17)^{(0)}.$$

The Leverrier coefficients depend on the Laplace functions, over the intermediary of the formulas /8

$$(2)^{(0)} = b_1^{(0)} + \frac{1}{2} b_2^{(0)} = \frac{1}{2} a c^{(1)},$$

$$(4)^{(0)} = \frac{1}{2} b_2^{(0)} + \frac{1}{8} b_4^{(0)},$$

$$(11)^{(0)} = -\frac{1}{2} a c^{(1)},$$

$$(12)^{(0)} = -a\left(c^{(1)} + 2c_1^{(1)} + \frac{1}{2} c_2^{(1)}\right),$$

$$(17)^{(0)} = \frac{3}{8} a^2 (2e^{(0)} + e^{(2)}).$$

In these expressions, Leverrier makes use of the notations

$$b_n^{(i)} = a^n \frac{d^n b^{(i)}}{da^n}, \quad c_n^{(i)} = a^n \frac{d^n c^{(i)}}{da^n}, \quad e_n^{(i)} = a^n \frac{d^n e^{(i)}}{da^n},$$

where $b^{(i)}$, $c^{(i)}$, $e^{(i)}$ are Laplace functions.

We will make use of the well-known expansions of these functions. Setting, for the time being,

$$k_n = \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)^2,$$

these expansions can be written as follows:

$$\frac{1}{2} b^{(0)} = 1 + \sum_{n=1}^{\infty} k_n a^{2n},$$

$$\frac{a}{2} c^{(1)} = \sum_{n=1}^{\infty} 2n(2n+1) k_n a^{2n},$$

$$\frac{a^2}{2} e^{(0)} = \frac{1}{9} \sum_{n=1}^{\infty} [2n(2n+1)]^2 k_n a^{2n}.$$

In view of this, it is easy to derive the formulas

$$N' = \frac{1}{16}\sum_{n=1}^{\infty} 2n(2n+1)(2n^2+5n+3)k_n a^{2n-1},$$ /9

$$N'' = \frac{1}{16}\sum_{n=1}^{\infty} 2n(2n+1)(2n^2+n+3)k_n a^{2n-1}$$

or else, in accordance with the expansion for $c^{(1)}$,

$$N' = \frac{1}{16}\left(3c^{(1)} + 2c_1^{(1)} + \frac{1}{4}c_2^{(1)}\right),$$

$$N'' = \frac{1}{16}\left(2c^{(1)} + c_1^{(1)} + \frac{1}{4}c_2^{(1)}\right).$$

This makes it obvious that the coefficients of $\rho_0'^2$ and $\rho_0''^2$ in the expression (2) for $\nu_1' + \nu_1''$ are positive.

Next, to demonstrate that the quantity (2) may vanish in the domain

$$0 < a < 1,$$

it is sufficient to prove that the quantity N may become positive within this domain. According to eqs.(1) and (3), this quantity N is linear with respect to e_0^2. The coefficient of e_0^2 is rational with respect to certain coefficients of the expansion of the perturbation function and remains finite since $F^{1,0,2,0}_{0,0,0,0}$ which enters the denominator does not vanish. The independent part of e_0^2 in the function N is

$$N_0 = F^{2,0,2,0}_{0,0,0,0} + F^{2,0,0,2}_{0,0,0,0}. \qquad (4)$$

Thus, to demonstrate that the quantity N can become positive, it is sufficient to prove that the quantity N_0 can become positive and is very large.

For this purpose, let us define the values of a or, rather, the values of $n_1 = a^{-3/2} (n_1 > 1)$ for which the function N_0 becomes infinite.

The quantities $F^{2,0,2,0}_{0,0,0,0}$ and $F^{2,0,0,2}_{0,0,0,0}$ are the coefficients of ρ_1^2 and of ρ_2^2 in the expansion of the function F_2, defined by eq.(6) in Section 3. Obviously, we /10 can put $e' = 0$. Then, in the expansion of the function S_1 given by eq.(5) of Section 3, we will have

$$\overline{m} = 0, \quad i_1 + i_2 = j_1 + j_2,$$

$$m_1 + m_2 = |i_1 + i_2| + 2k \qquad k \geq 0.$$

Thus, it is obvious that the wanted poles of the function N_0, considered as a function of n_1, will be

$$n_1 = \frac{q+1}{q} \qquad q = 1, 2, 3 \ldots \infty \qquad (5)$$

and

$$n_1 = \frac{q+2}{q} \qquad q = 1, 3, 5 \ldots \infty. \tag{6}$$

In addition, the poles (5) obviously are double whereas the poles (6) are single.

It is easy to see that

$$N_0 = 3x_1^{-4}(F^{1,0,1,0}_{-q,q+1,1,0})^2 \frac{1}{\left(n_1 - \frac{q+1}{q}\right)^2} + \cdots$$

is valid in the vicinity of the pole (5) and that

$$N_0 = -\frac{8}{q}\{(F^{1,0,2,0}_{-q,q+2,2,0})^2 + (F^{1,0,0,2}_{-q,q+2,0,2})^2\} \frac{1}{n_1 - \frac{q+2}{q}} + \cdots$$

applies in the vicinity of the pole (6).

Thus, the quantity N_0 is positive and very large when n_1 is close to $\frac{q+1}{q}$ and also when n_1 is slightly smaller than $\frac{q+2}{q}$. From this, it can be concluded that the quantity N is positive in the vicinity of the pole (5) up to a certain finite distance from this pole and that this quantity is also positive when n_1 is $< \frac{q+2}{q}$ up to a certain finite distance from this pole.

Let us put, in particular, q = 1. It is well known that the majority of /11 the minor planets is located within the domain

$$2 < n_1 < 3$$

limited by the two first poles (5) and (6). In certain parts of this domain, the quantity N is certainly positive; in these parts, we can select the values of $\rho_0'^1$ and $\rho_0''^2$ such that the quantity $\nu_1' + \nu_1''$ is canceled. From this, we can conclude that the existence of singular orbits is well possible.

Section 8.

Let us now return to the c al system of the secular inequalities, placed in the form of eq.(18) ion 3, i.e., to the equations

$$= \frac{dH}{d\eta'}, \quad \frac{d\eta'}{dt_1} = -\frac{dH}{d\xi'}, \tag{1}$$

$$\frac{d\xi''}{dt_1} = \frac{dH}{d\eta''}, \quad \frac{d\eta''}{dt_1} = -\frac{dH}{d\xi''}.$$

For H, we have the expansion

$$H = H^{(0)} + \mu H^{(1)} + \mu^2 H^{(2)} + \cdots \tag{2}$$

and, for $H^{(m)}$, the expression

$$H^{(m)} = \Sigma H_{j',j''}^{m;m' m''} \varrho'^{m'} \varrho''^{m''} \cos (j'\omega' + j''\omega'') \tag{3}$$

where

$$\begin{aligned} \xi' &= \varrho' \cos \omega', \quad \eta' = \varrho' \sin \omega', \\ \xi'' &= \varrho'' \cos \omega'', \quad \eta'' = \varrho'' \sin \omega''. \end{aligned} \tag{4}$$

The integers m', m'', j', j'' take only the values that satisfy the conditions

$$\begin{aligned} & m' + m'' \geq 2, \quad m'' = \text{even}, \\ & m' = |j'| + 2k', \quad m'' = |j''| + 2k'', \\ & 2m + 2 = |j' + j''| + m' + m'' + 2k \\ & \qquad = |j' + j''| + |j'| + |j''| + 2\tilde{k}, \end{aligned} \tag{5}$$

where k', k'', k, and $\tilde{k}$ are any nonnegative integers.

The system is of the type considered in Section 1. There are no variables corresponding to x_1, y_1; the variables ξ_k, η_k are denoted here by ξ', η', ξ'', η''. Finally, the quantities ν_0' and ν_0'' in the expression

$$H^{(0)} = -\frac{\nu_0'}{2} \varrho'^2 - \frac{\nu_0''}{2} \varrho''^2 \tag{6}$$

correspond to the quantities ν_k in Section 1.

We here have the identical relation

$$\nu_0' + \nu_0'' = 0. \tag{7}$$

It is possible to apply the reduction method given in Section 1. For this purpose, we start from the equation

$$H\left(\frac{dS}{d\eta'}, \eta', \frac{dS}{d\eta''}, \eta''\right) = H_*\left(\xi', \frac{dS}{d\xi'}, \xi'', \frac{dS}{d\xi''}\right). \tag{8}$$

The unknown functions H_* and S are expanded in the form

$$\begin{aligned} H_* &= H_*^{(0)} + \mu H_*^{(1)} + \mu^2 H_*^{(2)} + \cdots, \\ S &= S^{(0)} + \mu S^{(1)} + \mu^2 S^{(2)} + \cdots. \end{aligned}$$

We first have

$$H_*^{(0)} = H^{(0)} = -\frac{\nu_0'}{2} \varrho'^2 - \frac{\nu_0''}{2} \varrho''^2,$$

$$S^{(0)} = \xi'\eta' + \xi''\eta''.$$

By equating the coefficients of μ in the expansion of the two members of eq.(8), we will find the equation

$$\nu'_0 \frac{dS^{(1)}}{d\omega'} + \nu''_0 \frac{dS^{(1)}}{d\omega''} = H^{(1)} - H^{(1)}_*.$$

We then must select, for $H_*^{(1)}$, the part of the function $H^{(1)}$ that depends on ω' and ω'' only in the combination $\omega' + \omega''$. Given the conditions (5), it will be /13 found that $H_*^{(1)}$ thus becomes independent of ω' and ω''. We then find

$$H^{(1)}_* = H^{1,4,0}_{0,0}\varrho'^4 + H^{1,2,2}_{0,0}\varrho'^2\varrho''^2 + H^{1,0,4}_{0,0}\varrho''^4 + H^{1,2,0}_{0,0}\varrho'^2 + H^{1,0,2}_{0,0}\varrho''^2,$$

$$S^{(1)} = \Sigma' \frac{H^{1,m',m''}_{j',j''}}{j'\nu'_0 + j''\nu''_0}\varrho'^{m'}\varrho''^{m''}\sin(j'\omega' + j''\omega'')$$

where, in Σ', the terms with $j' = j''$ must be excluded.

By equating the coefficients of μ^2 in the expansion of the two members of eq.(8), we find

$$\nu'_0 \frac{dS^{(2)}}{d\omega'} + \nu''_0 \frac{dS^{(2)}}{d\omega''} = \bar{H}^{(2)} - H^{(2)}_*,$$

by putting, for abbreviation,

$$\bar{H}^{(2)} = H^{(2)} + \frac{dH^{(1)}}{d\xi'}\frac{dS^{(1)}}{d\eta'} + \frac{dH^{(1)}}{d\xi''}\frac{dS^{(1)}}{d\eta''} - \frac{\nu'_0}{2}\left(\frac{dS^{(1)}}{d\eta'}\right)^2 - \frac{\nu''_0}{2}\left(\frac{dS^{(1)}}{d\eta''}\right)^2$$
$$- \frac{dH^{(1)}_*}{d\eta'}\frac{dS^{(1)}}{d\xi'} - \frac{dH^{(1)}_*}{d\eta''}\frac{dS^{(1)}}{d\xi''} + \frac{\nu'_0}{2}\left(\frac{dS^{(1)}}{d\xi'}\right)^2 + \frac{\nu''_0}{2}\left(\frac{dS^{(1)}}{d\xi''}\right)^2.$$

For $H_*^{(2)}$, we must select the part of $\bar{H}^{(2)}$ that depends on ω' and ω'' only in the combination $\omega' + \omega''$. Now, in forming this part, it will be found that $H_*^{(2)}$ is independent of ω' and ω'' and that $H_*^{(2)}$ is just simply a polynomial of the third degree in ρ'^2 and ρ''^2. We will not write down its expression and also not the expression of the function $S^{(2)}$ which is a polynomial of the sixth degree in ξ', η', ξ'', η'' with certain symmetry properties.

It is necessary to investigate also the function $H_*^{(3)}$. By equating the coefficients of μ^3 in the two members of eq.(8), we find

$$\nu'_0 \frac{dS^{(3)}}{d\omega'} + \nu''_0 \frac{dS^{(3)}}{d\omega''} = \bar{H}^{(3)} - H^{(3)}_* \tag{9}$$

with the notation

$$\bar{H}^{(3)} = H^{(3)} + \frac{dH^{(2)}}{d\xi'}\frac{dS^{(1)}}{d\eta'} + \frac{dH^{(2)}}{d\xi''}\frac{dS^{(1)}}{d\eta''} - \frac{dH^{(2)}_*}{d\eta'}\frac{dS^{(1)}}{d\xi'} - \frac{dH^{(2)}_*}{d\eta''}\frac{dS^{(1)}}{d\xi''}$$ /14

$$
\begin{aligned}
&+\frac{dH^{(1)}}{d\xi'}\frac{dS^{(2)}}{d\eta'}+\frac{dH^{(1)}}{d\xi''}\frac{dS^{(2)}}{d\eta''}-\frac{dH_*^{(1)}}{d\eta'}\frac{dS^{(2)}}{d\xi'}-\frac{dH_*^{(1)}}{d\eta''}\frac{dS^{(2)}}{d\xi''}\\
&+\frac{1}{2}\frac{d^2H^{(1)}}{d\xi'^2}\left(\frac{dS^{(1)}}{d\eta'}\right)^2+\frac{d^2H^{(1)}}{d\xi'd\xi''}\frac{dS^{(1)}}{d\eta'}\frac{dS^{(1)}}{d\eta''}+\frac{1}{2}\frac{d^2H^{(1)}}{d\xi''^2}\left(\frac{dS^{(1)}}{d\eta''}\right)^2\\
&-\frac{1}{2}\frac{d^2H_*^{(1)}}{d\eta'^2}\left(\frac{dS^{(1)}}{d\xi'}\right)^2-\frac{d^2H_*^{(1)}}{d\eta'd\eta''}\frac{dS^{(1)}}{d\xi'}\frac{dS^{(1)}}{d\xi''}-\frac{1}{2}\frac{d^2H_*^{(1)}}{d\eta''^2}\left(\frac{dS^{(1)}}{d\xi''}\right)^2\\
&+\nu'_0\left\{\frac{dS^{(1)}}{d\xi'}\frac{dS^{(2)}}{d\xi'}\quad\frac{dS^{(1)}}{d\eta'}\frac{dS^{(2)}}{d\eta'}-\frac{dS^{(1)}}{d\xi''}\frac{dS^{( )}}{d\xi''}+\frac{dS^{(1)}}{d\eta''}\frac{dS^{(2)}}{d\eta''}\right\}.
\end{aligned}
$$

According to the rules given in Section 1, we must equate the quantity $H_*^{(3)}$ to the part of $H^{(3)}$ that depends on ω' and ω'' only in the combination $\omega' + \omega''$. For $H_*^{(3)}$, we find an expression of the following form:

$$
H_*^{(3)} = Q(\varrho'^2, \varrho''^2) + R(\varrho'^2, \varrho''^2)\varrho'^2\varrho''^2\cos(2\omega' + 2\omega''). \qquad (10)
$$

We will not give the expression of the polynomial Q which is of the fourth degree in ρ'^2 and ρ''^2. However, it is necessary for what follows to make a more detailed investigation of the polynomial R. In rather extensive (but well checked) calculations we found the following expression:

$$
R(\varrho'^2, \varrho''^2) = A + B\{N'\varrho'^2 + N''\varrho''^2 - N\}. \qquad (11)
$$

The constants N', N", N, A, and B are expressed by means of the coefficients $H_{j',j''}^{n,n',n''}$ which appear in the expressions (3) of the functions $H^{(n)}$. The three first of these constants are given by eq.(1) of Section 7. Below, we give the formulas for A and B:

$$
\begin{aligned}
A = {}& 2H_{2,2}^{3,2,2} + \frac{4}{\nu'_0}\{-H^{1,3,0}H_{1,2}^{2,1,2} - 2H_{1,0}^{1,1,2}H_{1,2}^{2,1,2}\\
&+ H_{1,-2}^{1,1,2}H^{2,3,0}_{\;\;0} + H_{2,0}^{1,2,0}H_{0,2}^{2,2,2} - H_{0,2}^{1,0,2}H_{2,0}^{2,2,2}\}\\
&+ \frac{8}{\nu_0'^2}\{H_{1,0}^{1,3,0}H_{1,0}^{1,1,2}H_{0,2}^{1,0,2} + 2(H_{1,0}^{1,1,2})^2H_{0,2}^{1,0,2} - 2H_{1,0}^{1,1,2}H_{1,-2}^{1,1,2}H_{2,0}^{1,2,0}\\
&+ H_{2,-2}^{1,2,2}(H_{2,0}^{1,2,0})^2 - H_{0,0}^{1,2,2}H_{2,0}^{1,2,0}H_{0,2}^{1,0,2} + H_{2,-2}^{1,2,2}(H_{0,2}^{1,0,2})^2\},\\
B = {}& \frac{4}{\nu_0'^2}\left\{\frac{2}{3}H_{1,0}^{1,3,0}H_{1,-2}^{1,1,2} - (H_{1,0}^{1,1,2})^2\right\}.
\end{aligned}
$$

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After thus having determined the function $H_*^{(3)}$, we can integrate eq.(9) which will yield $S^{(3)}$. This function will be a polynomial of the eighth degree in ξ', η', ξ'', η'' with certain symmetry properties. For our purpose, it is not necessary to give its expression here.

Evidently, we can continue in this manner and successively define the various functions $H_*^{(i)}$ and $S^{(i)}$.

In view of this and in accordance with the rules given in Section 1, we will introduce the new variables ξ'_*, η'_*, ξ''_*, η''_* by means of the canonical transformation

$$\xi' = \frac{dS(\xi'_*, \eta', \xi''_*, \eta'')}{d\eta'}, \quad \eta'_* = \frac{dS(\xi'_*, \eta', \xi''_*, \eta'')}{d\xi'_*},$$
$$\xi'' = \frac{dS(\xi'_*, \eta', \xi''_*, \eta'')}{d\eta''}, \quad \eta''_* = \frac{dS(\xi'_*, \eta', \xi''_*, \eta'')}{d\xi''_*}. \tag{12}$$

By solving for ξ', η', ξ'', η'', we will find that the differences $\xi' - \xi'_*$, $\eta' - \eta'_*$, $\xi'' - \xi''_*$, $\eta'' - \eta''_*$ vanish at $\mu = 0$ and can be expanded in powers of μ. The coefficients of μ^i in these series are polynomials of the degree $2i + 1$ in ξ'_*, η'_*, ξ''_*, η''_* with certain obvious symmetry properties.

The new variables satisfy the equations

$$\frac{d\xi'_*}{dt_1} = \frac{dH_*}{d\eta'_*}, \quad \frac{d\eta'_*}{dt_1} = -\frac{dH_*}{d\xi'_*},$$
$$\frac{d\xi''_*}{dt_1} = \frac{dH_*}{d\eta''_*}, \quad \frac{d\eta''_*}{dt_1} = -\frac{dH_*}{d\xi''_*}. \tag{13}$$

The new characteristic function $H_*(\xi'_*, \eta'_*, \xi''_*, \eta''_*)$ is obtained by merely writing ξ'_*, η'_*, ξ''_*, η''_* instead of ξ', η', ξ'', η'' in the expression of the function /16
$H_*(\xi', \eta', \xi'', \eta'')$ defined above.

It is easy to reduce the canonical system (13) to one degree of freedom. For this purpose, let us put

$$\xi'_* = \varrho'_* \cos \omega'_*, \quad \eta'_* = \varrho'_* \sin \omega'_*,$$
$$\xi''_* = \varrho''_* \cos \omega''_*, \quad \eta''_* = \varrho''_* \sin \omega''_*. \tag{14}$$

The characteristic function H_* of the system (13) depends on ω'_* and ω''_* only in the combination $\omega'_* + \omega''_*$. Consequently, instead of the variables

$$\xi'_*, \quad \xi''_*,$$
$$\eta'_*, \quad \eta''_*,$$

we can introduce first

$$\frac{1}{2}\varrho'^2_*, \quad \frac{1}{2}\varrho''^2_*,$$
$$\omega'_*, \quad \omega''_* \tag{15}$$

and then

$$\frac{1}{2}\varrho^2 = \frac{1}{2}\varrho'^2_*, \quad \frac{1}{2}\varkappa = \frac{1}{2}\varrho'^2_* - \frac{1}{2}\varrho''^2_*,$$
$$\omega = \omega'_* + \omega''_*, \quad -\omega''_*. \tag{16}$$

It is well known that these two transformations retain the canonical form.

The function H_*, expressed as a function of the new variables (16), does not depend on the second of the angular variables. Consequently, we will have the first integral

$$\varkappa = \text{const.}$$

In addition, we will have the canonical system with one degree of freedom

$$\frac{d\frac{1}{2}\varrho^2}{dt_1} = \frac{dH_*}{d\omega}, \qquad \frac{d\omega}{dt_1} = -\frac{dH_*}{d\frac{1}{2}\varrho^2}. \tag{17}$$

After integration, we finally obtain the variable ω_*'' by means of the equation /17

$$\frac{d\omega_*''}{dt_1} = \frac{dH_*}{d\frac{1}{2}\varkappa}. \tag{18}$$

We recall that H_* is given by the expansion

$$H_* = H_*^{(0)} + \mu H_*^{(1)} + \mu^2 H_*^{(2)} + \cdots.$$

We obviously have here

$$\begin{aligned}
&H_*^{(0)} = -\frac{n'_*}{2}\varkappa = \text{const.},\\
&H_*^{(1)} = -\frac{1}{2}(N' + N'')\varrho^4 + (N + N''\varkappa)\varrho^2 - H_{0,0}^{1,0,2}\varkappa + H_{0,0}^{1,0,4}\varkappa^2,\\
&H_*^{(2)} = \text{a polynomial of the third degree in } \rho^2 \text{ and } \varkappa\\
&H_*^{(3)} = Q(\varrho^2, \varrho^2-\varkappa) + R(\varrho^2, \varrho^2-\varkappa)\varrho^2(\varrho^2-\varkappa)\cos 2\omega,\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{19}$$

The polynomials Q and R are the same as those that enter eq.(10). The function $H_*^{(i)}$ is a polynomial in ρ^2 and $\varkappa$ of the degree i + 1, whose coefficients depend on ω. We can also state that $H_*^{(i)}$ is a polynomial in ρ^2 and $\rho^2(\rho^2 - \varkappa)\cos 2\omega$ with constant coefficients that depend on $\varkappa$.

Section 9.

In a geometric manner, we will now investigate the nature of the solutions of the canonical system (17) in Section 8.

To this end, we put

$$\xi = \varrho\cos\omega, \qquad \eta = \varrho\sin\omega,$$
$$\Phi = \frac{2(H_* - C)}{\mu(N' + N'')}, \tag{1}$$

$$t_2 = \mu (N' + N'') t_1 = \mu^2 (N' + N'') t,$$

where C is the value of the function H_* for $\rho = 0$. The new variables ξ and η satisfy the equations

$$2\frac{d\xi}{dt_2} = \frac{d\Phi}{d\eta}, \qquad 2\frac{d\eta}{dt_2} = -\frac{d\Phi}{d\xi}. \tag{2}$$

/18

This system has the integral

$$\Phi = h, \tag{3}$$

where h is an arbitrary constant.

Let us consider ξ and η as rectangular coordinates of one point of the plane. In a given solution of the system (2), the point ξ, η describes a certain curve in the family of contour lines defined by eq.(3) on varying there the parameter h. These contours are closed, cover the entire plane, and generally do not intersect. In addition, they are symmetric with respect to the axes. For each value of h, corresponding to a maximum or minimum value of the function Φ, the contour reduces to a point. These stationary points, as well as all the other singular points (double points, etc.) are obtained by solving the equations

$$\frac{d\Phi}{d\xi} = \frac{d\Phi}{d\eta} = 0.$$

First, we have the following solution:

$$\xi = \eta = 0.$$

The other singular points correspond to the tacnodes of the two curves

$$\frac{d\Phi}{d\varrho^2} = 0 \tag{4}$$

and

$$\frac{d\Phi}{d\omega} = 0. \tag{5}$$

The curve (4), which is single and closed, reduces, at $\mu = 0$, to the circle

$$\frac{dH_*^{(1)}}{d\varrho^2} \equiv -(N' + N'')\varrho^2 + N''x + N = 0. \tag{6}$$

The curve (5) is resolved into four different curves: /19

$$\text{the axis of } \xi \text{ where } \sin 2\omega = 0, \ \cos 2\omega = +1; \tag{7}$$

the axis of η where $\sin 2\omega = 0$, $\cos 2\omega = -1$; (8)

the circle $\rho^2 - \varkappa = 0$ (9)

and, finally, a single and closed curve which, at $\mu = 0$, is reduced to the circle

$$R(\varrho^2, \varrho^2-\varkappa) \equiv A + B\{(N'+N'')\varrho^2 - N''\varkappa - N\} = 0. \tag{10}$$

The concentric circles (6) and (10) do not coincide. Thus, outside of the origin, the only other singular points are the tacnodes of the curve (4) with the axis of ξ, with the axis of η, and with the circle $\rho^2 = \varkappa$.

Let us first study the singular points on the ξ axis. To obtain these points, it is sufficient to solve the equation

$$\left[\frac{d\Phi}{d\varrho^2}\right]_{\cos 2\omega = +1} = 0,$$

which yields the distance of the singular points from the origin. This equation has only one root ρ_ξ^2 since $H_*^{(1)}$ is of the second degree in ρ^2. Thus, we either will have two singular symmetric points on the ξ axis or else no such point, depending on whether $\rho_\xi^2 > 0$ or $\rho_\xi^2 < 0$.

Let us now define the singular points on the η axis. To find these points, it is sufficient to solve the equation

$$\left[\frac{d\Phi}{d\varrho^2}\right]_{\cos 2\omega = -1} = 0,$$

which has a single root ρ_η^2. This will yield two singular points on the η axis or else no such point, depending on whether $\rho_\eta^2 > 0$ or $\rho_\eta^2 < 0$.

The roots ρ_ξ^2 and ρ_η^2 can be expanded in powers of μ since the coefficients are polynomials in $\varkappa$. For $\mu = 0$, we have

$$\varrho_\xi^2 = \varrho_\eta^2 = \frac{N''\varkappa + N}{N' + N''}.$$

Thus, the equation $\rho_\xi^2 = 0$ has a single root $\varkappa = \varkappa_\xi'$ while the equation $\rho_\eta^2 = 0$ /20
has a single root $\varkappa = \varkappa_\eta'$. The quantities $\varkappa_\xi'$ and $\varkappa_\eta'$ can be expanded in powers of μ. For $\mu = 0$, we have

$$\varkappa_\xi' = \varkappa_\eta' = -\frac{N}{N''}.$$

In view of the fact that N' and N'' are positive, it is obvious that ρ_ξ^2 has the same sign as $\varkappa - \varkappa_\xi'$ and that ρ_η^2 has the same sign as $\varkappa - \varkappa_\eta'$.

Finally, let us investigate the singular points on the circle $\rho^2 = \varkappa$. To obtain these points, it is sufficient to determine the values of ω which satisfy the equation

$$\left[\frac{d\Phi}{d\varrho^2}\right]_{\varrho^2=\varkappa} = C. \qquad (11)$$

In view of the nature of the function $H_\varkappa$, discussed at the end of Section 8, this equation will have the form

$$f_0(\varkappa, \mu) + \mu^2 f_2(\varkappa, \mu) \cos 2\omega = 0, \qquad (11)$$

where f_0 and f_2 are certain power series in $\varkappa$ and μ. Obviously, we have

$$f_0(\varkappa, 0) = \left[\frac{dH_*^{(1)}}{d\varrho^2}\right]_{\varrho^2=\varkappa} = -N'\varkappa + N,$$

$$f_2(\varkappa, 0) = \varkappa R(\varkappa, 0) = \varkappa\{A + B(N'\varkappa - N)\}.$$

Equation (11) is satisfied by four real values of ω between zero and 2π, provided that the value of $\varkappa$ is located between two values $\varkappa_\xi''$ and $\varkappa_\eta''$ which are the unique roots, namely, $\varkappa_\xi'$ of the equation

$$\left[\frac{d\Phi}{d\varrho^2}\right]_{\substack{\varrho^2=\varkappa\\ \cos 2\omega=+1}} = 0,$$

and $\varkappa_\eta''$ of the equation

$$\left[\frac{d\Phi}{d\varrho^2}\right]_{\substack{\varrho^2=\varkappa\\ \cos 2\omega=-1}} = 0.$$

The quantities $\varkappa_\xi''$ and $\varkappa_\eta''$ can be expanded in powers of μ. Neglecting μ^2, we obtain /21

$$\varkappa_\xi'' = \varkappa_\eta'' = \frac{N}{N'}.$$

A comparison of the equations that define $\varkappa_\xi''$ and $\varkappa_\eta''$ with those that yield ρ_ξ^2 and ρ_η^2 will show that

$$\varrho_\xi^2 = \varkappa_\xi'' \qquad for \quad \varkappa = \varkappa_\xi'',$$

$$\varrho_\eta^2 = \varkappa_\eta'' \qquad for \quad \varkappa = \varkappa_\eta''.$$

Thus, for $\varkappa = \varkappa_\xi''$ resp. $\varkappa = \varkappa_\eta''$, the singular points on the circle $\rho^2 = \varkappa$ coincide with the singular points on the ξ axis resp. on the η axis.

Let us mention, in passing, that the differences $\varkappa_\xi' - \varkappa_\eta'$ and $\varkappa_\xi'' - \varkappa_\eta''$ are of the order of μ^2.

In speaking of the singular points of eqs.(2), attention should be drawn to the particular solution $\rho^2 = \varkappa$ which corresponds to the two-dimensional motion. The corresponding contour curve is a circle about the origin as center.

Considerable analogy exists between this solution and the other particular solution $\rho^2 = 0$. The investigated circle, in general, is an ordinary contour which does not osculate any of the other contour curves. This curve represents four double points only for the values of κ between κ''_ξ and κ''_η, whose arguments ω are determined by eq.(11). Let us also note that it is sufficient to investigate the contours located outside of the circle $\rho^2 = \kappa$. For the other curves, we would actually have $\rho''^2_* < 0$, which is impossible for the problem in question here.

Next, we will investigate the variation in the family of curves (3) with the values of the parameter κ.

Let us first note that the quantities N' and N" are always positive and that N may be either positive or negative depending on the value of the semi-major axis of the minor planet (see Section 7). In the following discussion, two cases will be differentiated, characterized by the sign of the quantity N.

Let us first assume that /22

$$N < 0,$$

such that κ'_ξ and κ'_η are > 0 while κ''_ξ and κ''_η are < 0. Let us vary κ, starting from very high negative values and proceeding toward very high positive values.

If $\kappa < 0$ and, a fortiori, $\kappa < \kappa'_\xi$ and κ'_η, no other singular points than the origin exist ($\xi = \eta = 0$). Here, the function Φ has an absolute maximum. (At infinite, Φ is always negative and very high.) The contour curves are arranged about the origin, one outside the other.

As soon as the parameter κ exceeds the value $\kappa = 0$, the contour circle $\rho^2 = \kappa$ will appear; for positive values of κ, it is sufficient to consider only the contour curves outside this circle. These exterior contour curves remain ordinary (i.e., without double points) for all positive values of κ. In fact, the singular points on the ξ axis and on the η axis separate from the origin at $\kappa = \kappa'_\xi$ and at $\kappa = \kappa'_\eta$, remaining always at the interior of the circle $\rho^2 = \kappa$. If this were not the case, the circle $\rho^2 = \kappa$ would stop being an ordinary contour curve for certain positive values of κ, which is impossible since κ''_ξ and κ''_η are negative in the case considered here. Thus, in the case in which $N < 0$, the contour curves that refer to our problem are arranged about the origin if $\kappa < 0$ and about the circle $\rho^2 = \kappa$ if $\kappa > 0$. On all these curves, there is no double point.

Let us then assume that

$$N > 0,$$

such that κ'_ξ and κ'_η are < 0, while κ''_ξ and κ''_η are > 0.

To fix the concept, let us suppose that $\kappa'_\xi < \kappa'_\eta$. If the opposite were true it would be sufficient to permute the ξ and η axes in the discussion given below.

If $\varkappa < \varkappa'_\xi$, there will be no other singular points than

$$(P_0) \qquad \xi = \eta = 0,$$

since ρ_ξ^2 and ρ_η^2 are negative. In this case, the function Φ has an absolute maximum in (P_0). The contour curves are arranged about this point without osculating. /23

If $\varkappa'_\xi < \varkappa < \varkappa'_\eta$, we will have $\rho_\xi^2 > 0$ and $\rho_\eta^2 < 0$. Then, the singular points will be (P_0) and

$$(P_\xi) \qquad \xi = \pm \varrho_\xi, \qquad \eta = 0.$$

Thus, Φ has a min-max in (P_0) and an absolute maximum in (P_ξ). There also is a contour curve with a crunode at the origin. This curve has the form

Fig.1

Here, we have a series of contour curves about each of the points (P_ξ) and, farther down, a series of curves surrounding the origin and the double-point contour curve. For the investigated values of $\varkappa$, the special planetary orbit which is approximately circular and corresponds to the point $\xi = \eta = 0$ obviously is unstable.

Next, if $\varkappa'_\eta < \varkappa < 0$, we have $\rho_\xi^2 > 0$ and $\rho_\eta^2 > 0$. Then, the singular points are (P_0), (P_ξ), and

$$(P_\eta) \qquad \xi = 0, \qquad \eta = \pm \varrho_\eta.$$

The function Φ has a relative minimum in (P_0), an absolute maximum in (P_ξ) and a min-max in (P_η). There also exists a contour curve with a crunode in the

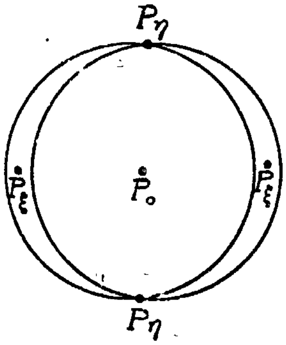

Fig.2

points (P_η). This curve has the form shown in Fig.2. The other contour curves are ordinary. We have a series of curves about each of the points (P_ξ) and, farther down, a series of curves about the point (P_0) as well as a series of curves surrounding the double-point contour curve. /24

As soon as the parameter $\varkappa$ exceeds the value $\varkappa = 0$, the contour circle $\rho^2 = \varkappa$ will appear. This is an ordinary contour curve as long as $0 < \varkappa < \varkappa''_\eta$. For these values of $\varkappa$, the general aspect of the contour curves still is that shown in Fig.2. However, only the contour curves outside of the circle $\rho^2 = \varkappa$ correspond to real orbits.

For $\varkappa = \varkappa''_\eta$, the contour circle $\rho^2 = \varkappa''_\eta$ passes through the singular point (P_η) and coincides with the interior part of the double-point contour curve.

If $\varkappa''_\eta < \varkappa < \varkappa''_\xi$, the double-point contour curve will have the form shown

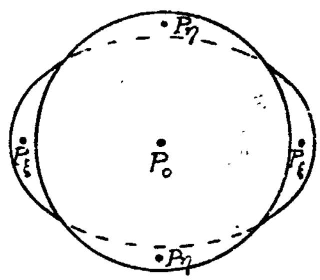

Fig.3

in Fig.3. One of its branches is the circle $\rho^2 = \varkappa$. The function Φ has maxima in (P_ξ) and in (P_η), a relative minimum in (P_0), and min-max in the four double points located on the circle $\rho^2 = \varkappa$. It is sufficient to consider only the contour curves surrounding each of the two points (P_ξ) as well as the curves farther toward the bottom which surround the entire double-point contour curve. For the investigated values of $\varkappa$, the plane planetary orbit, corresponding to the circle $\rho^2 = \varkappa$, obviously is unstable.

For $\varkappa = \varkappa''_\xi$, the contour circle $\rho^2 = \varkappa''_\xi$ passes through the singular point (P_ξ) and immediately thereafter separates from the exterior part of the double-point contour curve.

Finally, if $\varkappa > \varkappa''_\xi$, the double-point contour will have the form of the broken curves shown in Fig.4. The circle $\rho^2 = \varkappa$ envelops this curve completely; outside of this circle, the contours (the only ones to be considered) are ordinary and concentric. /25

On all of the ordinary contour curves, ξ and η are periodic functions of t_2. The period relative to t has at least the order of μ^{-2}, i.e., is very large. As soon as an ordinary contour curve more and more approaches a double-point contour curve, the period will tend toward infinite. On the double-point contour curves, the point ξ, η indefinitely approaches one of the crunodes when t increases or decreases toward infinite.

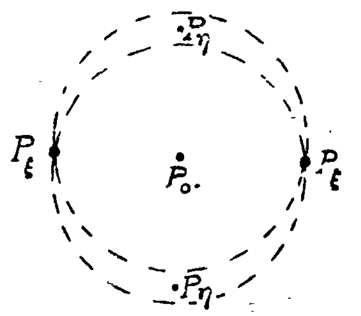

Fig.4

On the ordinary contours, located higher than the min-max of Φ [i.e., or the curves surrounding the absolute maximum of the function Φ in (P_ξ)], the argument ω varies between two limits. For these solutions, it thus has a sort of libration. On the contours, located lower than the min-max of the function Φ, the argument ω has a mean motion.

For the existence of libration, it is necessary and sufficient at first that the double-point contour curve exist , which is expressed by the conditions

$$N > 0, \quad x'_\xi < x < x''_\xi,$$

and then, that the value of h be greater than the min-max values of the function Φ. Moreover, on the singular curve, the difference of the two values of ρ is generally and at least of the order of μ (see the following Section). Thus, cases of libration occur quite rarely in nature. /26

In Part I of this research (pp.2 and 36), we differentiated between regular orbits and singular orbits. For the former, the quantity

$$(N' + N'')\rho^2 - N''x - N$$

is comparable to unity; for the latter, this quantity is of the order of $\sqrt{\mu}$ or smaller. It is now a question which are the contour curves in the discussion of Section 9 that correspond to singular orbits. Evidently, these contour curves (projected onto the plane of the ξ, η) are located in the vicinity of the double-point curve up to a distance of the order of $\sqrt{\mu}$.

Section 10.

We have geometrically investigated the nature of the solutions of the system (17) in Section 8. Below, we will give the computation of these solutions.

By using the definitions of the function Φ and of the variable t_2, given at the beginning of Section 9, eqs.(17) of Section 8 can be written as follows:

$$\frac{d\varrho^2}{dt_2} = \frac{d\Phi}{d\omega}, \qquad \frac{d\omega}{dt_2} = -\frac{d\Phi}{d\varrho^2}. \tag{1}$$

The function $\Phi(\rho^2, \omega)$ can be expanded in the form of

$$\Phi = \Phi^{(0)} + \mu\Phi^{(1)} + \mu^2\Phi^{(2)} + \cdots. \tag{2}$$

Making use of the notation

$$\bar{\varrho}^2 = \frac{N + N''\varkappa}{N' + N''}, \tag{3}$$

we will obtain

$\Phi^{(0)} = -\varrho^4 + 2\bar{\varrho}^2\varrho^2,$

$\Phi^{(1)} =$ a polynomial of the third degree in ρ^2 and $\varkappa$, /27

$$\Phi^{(2)} = \left\{\frac{2A}{N' + N''} + 2B(\varrho^2 - \bar{\varrho}^2)\right\}\varrho^2(\varrho^2 - \varkappa)\cos 2\omega \tag{4}$$

$+$ a polynomial of the fourth degree in ρ^2 and $\varkappa$

....

In general, $\Phi^{(i)}$ is a polynomial with respect to the quantities

$$\varkappa, \ \varrho^2 \ \text{and} \ \varrho^2(\varrho^2 - \varkappa)\cos 2\omega.$$

For integrating the system (1), we will use the Jacobi method. Let, thus, $S(\omega)$ be any function satisfying the equation

$$\Phi\left(\frac{dS}{d\omega}, \omega\right) = h, \tag{5}$$

where h is an arbitrary constant. Then, the general solution of the system (1) is obtained by means of the relations

$$\rho^2 = \frac{dS}{d\omega}, \quad \frac{dt_2}{d\omega} = -\frac{d^2S}{d\omega dh}. \tag{6}$$

To solve eq.(5) in a general manner, we will throw the function $h - \Phi$ into the form

$$h-\Phi(\varrho^2,\omega)=(\varphi^{(0)}+\mu\varphi^{(1)}+\mu^2\varphi^{(2)}+\cdots)(1+\mu\psi^{(1)}+\mu^2\psi^{(2)}+\cdots),$$

where $\varphi^{(1)}$ and $\psi^{(1)}$ are polynomials in ρ^2 with coefficients that depend on $\varkappa$ and on cos $2w$. These polynomials must satisfy the identities

$$\varphi^{(0)}=h-\Phi^{(0)},$$
$$\varphi^{(1)}+\varphi^{(0)}\psi^{(1)}=-\Phi^{(1)},$$
$$\varphi^{(2)}+\varphi^{(1)}\psi^{(1)}+\varphi^{(0)}\psi^{(2)}=-\Phi^{(2)},$$
$$\cdots\cdots\cdots\cdots\cdots$$

For $\varphi^{(1)}$, $\varphi^{(2)}$, ..., we can select polynomials of the first degree in ρ^2. We will then put

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$$\varphi^{(0)}=\varphi_0^{(0)}-2\varphi_1^{(0)}\varrho^2+\varphi_2^{(0)}\varrho^4,$$
$$\varphi^{(1)}=\varphi_0^{(1)}-2\varphi_1^{(1)}\varrho^2,$$
$$\varphi^{(2)}=\varphi_0^{(2)}-2\varphi_1^{(2)}\varrho^2,$$
$$\cdots\cdots\cdots\cdots\cdots$$

The coefficients of the polynomial $\varphi^{(0)}$ are, quite simply,

$$\varphi_0^{(0)}=h,\qquad \varphi_1^{(0)}=\bar{\varrho}^2,\qquad \varphi_2^{(0)}=1.$$

Since $\varphi_2^{(0)}$ is independent of h, $\varkappa$, and cos $2w$, it is obvious that the coefficients of the various powers of ρ^2 in $\psi^{(1)}$ and $\varphi^{(1)}$ are polynomials in h, $\varkappa$, cos $2w$. In addition, the coefficients of $\psi^{(1)}$ and of $\varphi^{(1)}$ are independent of cos $2w$ since $\Phi^{(1)}$ is independent of this. Finally, $\psi^{(2)}$ and $\varphi^{(2)}$ are linear in cos $2w$ as is the case for $\Phi^{(2)}$.

In view of this, eq.(5) can be written as follows:

$$\left(\frac{dS}{d\omega}\right)^2-2\varphi_1\frac{dS}{d\omega}+\varphi_0=0. \tag{7}$$

Here, φ_0 and φ_1 are functions of w given by the expansions

$$\varphi_0=\varphi_0^{(0)}+\mu\varphi_0^{(1)}+\mu^2\varphi_0^{(2)}+\cdots,$$
$$\varphi_1=\varphi_1^{(0)}+\mu\varphi_1^{(1)}+\mu^2\varphi_1^{(2)}+\cdots.$$

Now, by putting

$$D=\varphi_1^2-\varphi_0,$$

the correlation between ρ^2 and w can be written as

$$\varrho^2=\frac{dS}{d\omega}=\varphi_1\pm\sqrt{D}. \tag{8}$$

Finally, the correlation between t_2 and ω becomes

$$\frac{dt_2}{d\omega} = -\frac{d\varphi_1}{dh} \mp \frac{1}{2\sqrt{D}}\frac{dD}{dh}, \tag{9}$$

It is of interest to investigate the function D in more detail. We can expand this function in powers of μ, by putting /29

$$D = D^{(0)} + \mu D^{(1)} + \mu^2 D^{(2)} + \cdots.$$

The quantities $D^{(i)}$ are polynomials in h, $\varkappa$, and cos 2ω. Here, $D^{(0)}$ and $D^{(1)}$ are independent of ω. We have, specifically,

$$D^{(0)} = \bar{\varrho}^4 - h, \quad \frac{dD^{(0)}}{dh} = -1.$$

Instead of giving the rather complicated expressions for $D^{(1)}$, $^{(2)}$, $D^{(3)}$, ...; we prefer to resolve the function D into two factors and to investigate specifically the expansion of the particular factor which might cancel out.

For this purpose, we consider the equation

$$D = 0, \tag{10}$$

where h is assumed as being unknown. For small values of μ, eq.(10) admits of a single root which can be expanded in powers of μ. It is easy to form the expansion of this root. In fact, it is here a question of finding the value of h, expressed as a function of ω, for which eq.(5) has a double root. From this it follows that

$$h = \Phi(\tilde{\varrho}^2, \omega), \tag{11}$$

where $\tilde{\rho}^2$ is the root of the equation

$$\frac{d\Phi}{d\varrho^2} = 0,$$

which can be written in the following form

$$2(\varrho^2 - \bar{\varrho}^2) - \mu\frac{d\Phi^{(1)}}{d\varrho^2} - \mu^2\frac{d\Phi^{(2)}}{d\varrho^2} - \cdots = 0.$$

For small values of μ, this equation has a single root $\tilde{\rho}^2$. The root in question, as well as any function of this root, can be readily obtained by means of a Lagrange series. This will yield /30

$$\Phi(\tilde{\varrho}^2, \omega) = \Phi(\bar{\varrho}^2, \omega) + \tag{12}$$
$$+ \sum_{s=0}^{\infty} \frac{1}{2^{s+1}\,\overline{s+1}\,!}\frac{d^s}{(d\bar{\varrho}^2)^s}\left(\mu\frac{d\Phi^{(1)}}{d\bar{\varrho}^2} + \mu^2\frac{d\Phi^{(2)}}{d\bar{\varrho}^2} + \cdots\right)^{s+2}.$$

In view of the fact that the root (11) of eq.(10) is given, it is possible to resolve the function D into two factors. We will then put

$$D = \Delta f.$$

The first factor has the expression

$$\Delta = \Phi(\bar{\varrho}^2, \omega) - h. \tag{13}$$

By setting $\mu = 0$, we have $\Delta = \bar{\rho}^4 - h = D^{( )}$ from which it follows that the second factor f reduces to unity at $\mu = 0$. The quantities Λ and f can be expanded in powers of μ. Their expansions are written in the form of

$$\Delta = \Delta^{(0)} + \mu\Delta^{(1)} + \mu^2\Delta^{(2)} + \cdots,$$
$$f = 1 + \mu f^{(1)} + \mu^2 f^{(2)} + \cdots.$$

It is easily found that

$$\Delta^{(0)} = \Phi^{(0)}(\bar{\varrho}^2) - h = \bar{\varrho}^4 - h,$$
$$\Delta^{(1)} = \Phi^{(1)}(\bar{\varrho}^2),$$
$$\Delta^{(2)} = \frac{1}{2}\left(\frac{d\Phi^{(1)}}{d\bar{\varrho}^2}\right)^2 + \Phi^{(2)}(\bar{\varrho}^2) = \frac{2A}{N' + N''}\bar{\varrho}^2(\bar{\varrho}^2 - \varkappa)\cos 2\omega$$
$$+ \text{ a polynomial of the fourth degree in } \varkappa. \tag{14}$$
$$\Delta^{(3)} = \frac{3}{8}\left(\frac{d\Phi^{(1)}}{d\bar{\varrho}^2}\right)^2 \frac{d^2\Phi^{(1)}}{(d\bar{\varrho}^2)^2} + \frac{d\Phi^{(1)}}{d\bar{\varrho}^2}\frac{d\Phi^{(2)}}{d\bar{\varrho}^2} + \Phi^{(3)}(\bar{\varrho}^2),$$

. . . .

It is not necessary to give the expressions for the functions $f^{(i)}$. It is sufficient to note that $f^{(i)}$ is a polynomial in h, $\varkappa$, and cos 2ω as well as that /31
$f^{(1)}$ is independent of ω.

We will also require the trigonometric expansion of Λ. Let

$$\Delta = \Delta_0 + \Delta_2 \cos 2\omega + \Delta_4 \cos 4\omega + \cdots \tag{15}$$

be this development. On the basis of eqs.(14), it is easy to form the coefficients $\Lambda_0 + h$, Λ_2, Λ_4, ... which can be expanded in powers of μ with coefficients that are polynomials in $\varkappa$, independent of h. The coefficient Δ_2 is of the order of μ^2. We have verified that the function $H_*^{(4)}$ contains no term in cos 4ω. From this it follows that $\Phi^{(3)}$ and $\Delta^{(3)}$ are linear in cos 2ω and that Δ_4 is of the order of μ^4.

Later, we will consider the root $\bar{h}(\varkappa)$ defined by the equation

$$\Delta_0 = 0,$$

as well as the two roots $\varkappa'$ and $\varkappa''$ of the equation

$$\Delta_3 = 0.$$

Below, we give the first terms of their expansions in powers of μ:

$$\bar{h}(x) = \bar{\varrho}^4 + \mu\Phi^{(1)}(\bar{\varrho}^2) + \cdots,$$
$$x' = -N:N'' + \cdots,\quad x'' = N:N' + \cdots.$$

Thus, we will identically have

$$\Delta_0 = \bar{h}(x) - h,$$
$$\Delta_2 = (x - x')(x - x'')\mu^2\nabla_2,$$

where ∇_2 can be expanded in powers of μ and reduces, at $\mu = 0$, to a constant which is independent of $\varkappa$ and $\neq 0$.

For convenient values of the constants $\varkappa$ and h, the coefficients Λ_0 and Λ_2 take any values and, specifically, values as small as desired.

To form the general solution of the canonical system (1), we will start /32 from eqs.(8) and (9). It becomes necessary to differentiate several cases, depending on the relative magnitude of the coefficients in the trigonometric expansion (15) of the function Λ.

Section 11.

In this Section, we will assume that the ratios

$$\frac{\Delta_{2i}}{\Delta_0} \qquad (i = 1, 2, 3, \ldots)$$

are comparable, in magnitude, with μ or are smaller.

We will now define certain typical cases which may occur here and to which correspond certain limitations imposed on the parameters $\varkappa$ and h.

Thus, let Λ_0 be of the order of μ^{ε}. The various typical cases, to be discussed here, correspond to the various values of the whole number ε.

For $\epsilon = 0$, the quantity $\varkappa$ may remain arbitrary; $h - \bar{h}(\varkappa)$ must be comparable, in magnitude, to unity. This is the case of so-called regular planets, treated in Part I of this research.

For $\epsilon = 1$, the quantity $\varkappa$ can take any value, but $h - \bar{h}(\varkappa)$ must be of the order of μ.

In these two cases, the coefficients Λ_4, ..., are at least of the order of μ^4.

For $\epsilon = 2, 3, 4, 5$, one or the other of the quantities $\varkappa - \varkappa'$ and $\varkappa - \nu''$ must be of the order of $\mu^{\epsilon-1}$ or smaller; the quantity $h - \overline{h}(\varkappa)$ must be of the order of μ^3.

In these four cases, the coefficients Δ_4, ... are at least of the order of μ^6 since the various terms of these coefficients include one or the other of the three factors $\mu^4\overline{\rho}^4(\overline{\rho}^2 - \varkappa)^2$, $\mu^5\overline{\rho}^2(\overline{\rho}^2 - \varkappa)$, μ^6, which all are of the order of μ^6.

It is impossible to have $\epsilon \geq 6$ since then the ratio $\Delta_4:\Delta_0$ would no longer be small.

Obviously, it is sufficient to consider in all cases only the values of h which are $< \overline{h}(\varkappa)$, since the function D must be positive.

We will investigate the six mentioned typical cases as a unit. /33

In all these cases, the quantity $\sqrt{D:\mu^\epsilon}$ can be expanded in powers of μ. Thus, the function ρ^2 can be expanded in powers of μ for $\epsilon = 0, 2, 4$ and in powers of $\sqrt{\mu}$ for $\epsilon = 1, 3, 5$. The successive terms of the series are polynomials in $\cos 2w$. We can proceed further and assert that the functions

$$\varrho \text{ and } \sqrt{\varrho^2 - \varkappa}$$

can be expanded in the same manner.

To demonstrate this, we will write, on the one hand, the two roots of eq.(8) in various manners, depending on the sign of φ_1 or of $\varphi_1 - \varkappa$. Thus, we put

$$\varrho^2 = \begin{cases} \varphi_1 + \sqrt{D}, \\ \dfrac{\varphi_0}{\varphi_1 + \sqrt{D}} \end{cases} \qquad if\ \varphi_1 > 0,$$

$$\varrho^2 = \begin{cases} \dfrac{-\varphi_0}{-\varphi_1 + \sqrt{D}}, \\ \varphi_1 - \sqrt{D} \end{cases} \qquad if\ \varphi_1 < 0;$$

as well as

$$\varrho^2 - \varkappa = \begin{cases} \varphi_1 - \varkappa + \sqrt{D}, \\ \dfrac{\varkappa^2 - 2\varphi_1\varkappa + \varphi_0}{\varphi_1 - \varkappa + \sqrt{D}} \end{cases} \qquad if\ \varphi_1 > \varkappa,$$

$$\varrho^2 - \varkappa = \begin{cases} \dfrac{-(\varkappa^2 - 2\varphi_1\varkappa + \varphi_0)}{-(\varphi_1 - \varkappa) + \sqrt{D}}, \\ \varphi_1 - \varkappa - \sqrt{D} \end{cases} \qquad if\ \varphi_1 < \varkappa.$$

In these formulas, $\sqrt{D}$ always denotes the positive root.

On the other hand, we note that the functions

$$\sqrt{\varphi_0} \text{ and } \sqrt{\varkappa^2 - 2\varphi_1\varkappa + \varphi_0}$$

are always expandable in powers of μ. In fact, φ_0 is divisible by h since one of the roots ρ^2 reduces to zero with h. Similarly, the function $\varkappa^2 - 2\varphi_1\varkappa + \varphi_0$ is divisible by $h - \Phi(\varkappa, \omega)$ since one of the roots ρ^2 reduces to $\varkappa$ as soon as h assumes the constant value $\Phi(\varkappa, \omega)$. Finally the quotients /34

$$\frac{\varphi_0}{h} \text{ and } \frac{\varkappa^2 - 2\varphi_1\varkappa + \varphi_0}{h - \Phi(\varkappa, \omega)}$$

reduce to unity for $\mu = 0$.

Let us now return to eq.(9), by writing it in the following manner:

$$\frac{d\mu^{\frac{\varepsilon}{2}} t_2}{d\omega} = -\mu^{\frac{\varepsilon}{2}} \frac{d\varphi_1}{dh} \mp \frac{1}{2} \sqrt{\frac{\mu^{\varepsilon}}{D}} \frac{dD}{dh} = P(\omega). \tag{1}$$

The function $P(\omega)$ can be expanded in powers of μ for $\varepsilon = 0, 2, 4$ and in powers of $\sqrt{\mu}$ for $\varepsilon = 1, 3, 5$. The various terms of the expansion are finite trigonometric cosine series of multiples of the argument 2ω. Let $[P]$ be the mean value of the function $P(\omega)$. The constant $[P]$ is comparable, in magnitude, to unity and can be expanded in powers of μ for $\varepsilon = 0, 2, 4$, and in powers of $\sqrt{\mu}$ for $\varepsilon = 1, 3, 5$.

We will put (γ being an arbitrary constant)

$$w = \frac{\mu^{\frac{\varepsilon}{2}} t_2}{[P]} + \gamma = \mu^{2+\frac{\varepsilon}{2}} \frac{N' + N''}{[P]} t + \gamma, \tag{2}$$

$$\tilde{P}(\omega) = \frac{-1}{\mu[P]} \int_0^{\omega} (P(\omega) - [P])\, d\omega. \tag{3}$$

The correlation between w and ω will then be

$$\omega - w - \mu\tilde{P}(\omega) = 0. \tag{4}$$

The function $\tilde{P}(\omega)$ can be expanded in powers of μ for $\varepsilon = 0, 2, 4$ and in powers of $\sqrt{\mu}$ for $\varepsilon = 1, 3, 5$. The various terms of the expansion are finite trigonometric sine series of multiples of the argument 2ω. Finally, $\tilde{P}(\omega)$ is of the order of μ if $\varepsilon = 0$; and of the order of $\mu^0 = 1$ if $\varepsilon = 1, 2, 3, 4, 5$. /35

The relation (4) can be solved by the Lagrange formula. Let $\Pi(\omega)$ be any function of ω. We then have

$$\Pi(\omega) = \Pi(w) + \sum_{s=0}^{\infty} \frac{\mu^{s+1}}{s+1!} \frac{d^s}{dw^s} \left\{ \frac{d\Pi(w)}{dw} \tilde{P}(w)^{s+1} \right\}. \tag{5}$$

We can replace $\Pi(\omega)$ by any of the functions

$$\omega, \quad \varrho, \quad \sqrt{\varrho^2 - \varkappa}, \quad \xi = \varrho\cos\omega, \quad \eta = \varrho\sin\omega.$$

Thus, it becomes possible to calculate the solution of the system (1) of Section 10 for the typical cases considered in this Section.

Section 12.

Let us now assume that the ratio

$$\frac{\Delta_0}{\Delta_2}$$

is not large and that, in addition, all the ratios

$$\frac{\Delta_4}{\Delta_2}, \frac{\Delta_6}{\Delta_2}, \ldots$$

are of the order of μ or smaller.

Let us consider the typical cases in which Δ_2 is of the order of μ^{ε} with ϵ being a whole number.

For $\epsilon = 2, 3, 4, 5$, the quantity $\varkappa - \varkappa'$ or $\varkappa - \varkappa''$ must be of the order of $\mu^{\varepsilon-2}$ and, at the same time, $h - h(\varkappa)$ must be of the order of μ^{ε} or smaller. Then, Δ_4 is of the order of μ^4 for $\epsilon = 2$ and of the order of μ^6 for $\epsilon = 3, 4, 5$. The coefficients Δ_6, ... are still smaller.

It is impossible to have $\epsilon \geq 6$ since then the ratio

$$\frac{\Delta_4}{\Delta_2}$$

would no longer be small.

In all four cases ($\epsilon = 2, 3, 4, 5$) we can throw the function D into the form /36

$$D = \mu^{\varepsilon}(\sigma - \sin^2\omega)g = \mu^{\varepsilon}\sigma\left(1 - \frac{1}{\sigma}\sin^2\omega\right)g,$$

where σ is an arbitrary constant which can replace h, while g is a function of ω differing from zero and expandable in powers of μ. Let

$$g = g^{(0)} + \mu g^{(1)} + \mu^2 g^{(2)} + \cdots$$

be the expansion of the function g. The quantities $g^{(i)}$ are polynomials in cos 2ω; the quantity $g^{(0)}$ always is a constant while $g^{(1)}$ depends on cos 2ω only for $\epsilon = 5$.

We do not know the sign of the constant $g^{(0)}$. For $\epsilon = 2$, $g^{(0)}$ has the same sign as the coefficient A of which we know only the analytical expression (see p.57). However, to fix the concept, we will assume that $g_0 > 0$. If this were not the case, it would be sufficient below to consider $1 - \sigma$ and $\omega + \frac{\pi}{2}$ instead of σ and ω.

Since D must never become negative, it is sufficient to consider the positive values of σ. In view of this, two cases must be differentiated, depending on the value of σ.

<u>First case</u>: $\sigma > +1.$

In this case, it is convenient to put

$$k^2 = \frac{1}{\sigma}$$

and to replace ω by the variable

$$u = \int_0^{\omega} \frac{d\omega}{\sqrt{1 - k^2 \sin^2 \omega}}.$$

This will yield

$$\sin \omega = \operatorname{sn} u,$$ /37

$$\cos \omega = \operatorname{cn} u,$$

$$\sqrt{1 - k^2 \sin^2 \omega} = \operatorname{dn} u.$$

We will denote by K the complete elliptical integral of the first kind relative to the modulus k, i.e.,

$$K = \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{1 - k^2 \sin^2 \omega}}.$$

It is well known that

$$\operatorname{sn}(u + 2K) = -\operatorname{sn} u,$$

$$\operatorname{cn}(u + 2K) = -\operatorname{cn} u,$$

$$\operatorname{dn}(u + 2K) = \operatorname{dn} u.$$

In view of this, we can write eqs.(8) and (9) of Section 10 in the following manner:

$$\rho^2 = \varphi_1 \pm \mu^{\frac{\varepsilon}{2}} \sqrt{g}\, k^{-1} \operatorname{dn} u, \tag{1}$$

$$\frac{d\mu^{\frac{\varepsilon}{2}} t_2}{du} = -\mu^{\frac{\varepsilon}{2}} \frac{d\varphi_1}{dh} \operatorname{dn} u \mp \frac{k}{2\sqrt{g}} \frac{dD}{dh} = Q(u). \tag{2}$$

Second case: $0 < \sigma < +1.$

It is convenient to put

$$k^2 = \sigma, \quad \sin\omega = k\sin\omega'$$

and to replace ω by the variable

$$u = \int_0^{\omega} \frac{d\omega}{\sqrt{k^2 - \sin^2\omega}} = \int_0^{\omega'} \frac{d\omega'}{\sqrt{1 - k^2\sin^2\omega'}}.$$

This will yield

$$\sin\omega = k\,\mathrm{sn}\,u,$$ /38

$$\cos\omega = \mathrm{dn}\,u,$$

$$\sqrt{k^2 - \sin^2\omega} = k\,\mathrm{cn}\,u.$$

Let K be the complete integral of the first kind, corresponding to the modulus k. (It should not lead to confusion that in both cases we use exactly the same symbols k^2, K, u, Q(u), v, w, ..., for denoting certain analogous but different quantities.)

In this second case, eqs.(8) and (9) of Section 10 take the form

$$\varrho^2 = \varphi_1 + \mu^{\frac{\varepsilon}{2}}\sqrt{g}\,k\,\mathrm{cn}\,u, \tag{1'}$$

$$\frac{d\mu^{\frac{\varepsilon}{2}}t_2}{du} = -\mu^{\frac{\varepsilon}{2}}\frac{d\varphi_1}{dh}k\,\mathrm{cn}\,u - \frac{1}{2\sqrt{g}}\frac{dD}{dh} = Q(u). \tag{2'}$$

The first and the second case can be treated together.

Let us note first that φ_1 and $\varphi_1 - \varkappa$ are of the order of μ^0 for $\varepsilon = 2$. For $\varepsilon = 3, 4$, or 5, the quantity φ_1 is of the order of μ and $\varphi_1 - \varkappa$ is of the order of μ^0 if $\varkappa - \varkappa'$ is of the order of $\mu^{\varepsilon-2}$; conversely, φ_1 is of the order of μ^0 and $\varphi_1 - \varkappa$ of the order of μ if it is $\varkappa - \varkappa''$ that is of the order of $\mu^{\varepsilon-2}$. In view of this, it is obvious that the functions

$$\varrho, \quad \sqrt{\varrho^2 - \varkappa} \text{ and } Q(u)$$

can be expanded in powers of μ or $\sqrt{\mu}$, depending on whether ε is even or odd. The various terms of these expansions are polynomials; in the first case ($\sigma > 1$) with respect to k, k^{-1}, and dn u; in the second case ($\sigma < 1$) with respect to k and k cn u.

Let us find the correlation between u and t.

Let [Q] be the mean value of the periodic function Q(u). The constant [Q] is comparable in magnitude with unity and can be expanded in powers of μ resp. $\sqrt{\mu}$, depending on whether ϵ is even or odd.

It is convenient to introduce the notations

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$$v = \frac{\mu^{\frac{\epsilon}{2}} t_7}{[Q]} + \text{const.} = \mu^{2+\frac{\epsilon}{2}} \frac{N' + N''}{[Q]} t + \text{const.}, \tag{3}$$

$$\tilde{Q}(u) = \frac{-1}{\mu [Q]} \int_0^u (Q(u) - [Q])\, du. \tag{4}$$

The correlation between u and v is written as

$$u - v - \mu \tilde{Q}(u) = 0. \tag{5}$$

The function $\tilde{Q}(u)$ can be expanded in powers of μ or $\sqrt{\mu}$. The various terms of this series are odd periodic functions of μ, with the period 2K in the first case ($\sigma > 1$) and 4K in the second case ($\sigma < 1$). Finally, $\tilde{Q}(u)$ is of the order of μ if $\epsilon = 2, 3, 4$ and of the order of $\mu^0 = 1$ if $\epsilon = 5$.

The relation (5) can be solved by the Lagrange method. Let $\Pi(u)$ be any function of u. We will then have

$$\Pi(u) = \Pi(v) + \sum_{s=0}^{\infty} \frac{\mu^{s+1}}{s+1!} \frac{d^s}{dv^s} \left\{ \frac{d\Pi(v)}{dv} (\tilde{Q}(v))^{s+1} \right\}. \tag{6}$$

Let us study the nature of the function $\tilde{Q}(u)$ in more detail. We have seen above that the various terms of the expansion of the function Q(u) in powers of μ or $\sqrt{\mu}$ are polynomials in k, k^{-1}, and dn u in the first case and polynomials with respect to k and k cn u in the second case. To form the function $\tilde{Q}(u)$, given by eq.(4), we will have to consider, in the first case, the integrals

$$D_{2s+1} = \int_0^u \mathrm{dn}^{2s+1} u\, du, \quad D_{2s} = \int_0^u \mathrm{dn}^{2s} u\, du,$$

and, in the second case, the integrals

$$C_{2s+1} = k^{2s+1} \int_0^u \mathrm{cn}^{2s+1} u\, du, \quad C_{2s} = k^{2s} \int_0^u \mathrm{cn}^{2s} u\, du.$$

Denoting by $L_\nu(\mathrm{dn}^2 u)$ certain polynomials in k^2 and $\mathrm{dn}^2 u$ and by δ_ν and δ'_ν certain polynomials in k^2, we will have for the first case ($\sigma > 1$) /40

$$D_{2s+1} = \int_0^\omega (1 - k^2 \sin^2 \omega)^s\, d\omega$$

$$= k^2 \sin \omega \cos \omega\, L_{2s+1} (1 - k^2 \sin^2 \omega) + \delta_{2s+1} \int_0^\omega d\omega$$

$$= k^2 \operatorname{sn} u \operatorname{cn} u\, L_{2s+1}(\operatorname{dn}^2 u) + \delta_{2s+1} D_1,$$

$$D_{2s} = \int_0^{\omega} (1 - k^2 \sin^2 \omega)^{\frac{2s-1}{2}} d\omega = \frac{k^2}{2s-1} \sin \omega \cos \omega (1 \qquad \omega)^{\frac{2s-3}{2}}$$

$$+ \frac{2s-2}{2s-1}(2-k^2) D_{2s-2} - \frac{2s-3}{2s-1}(2-k^2) D_{2s-4}$$

$$= k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\, L_{2s}(\operatorname{dn}^2 u) + \delta_{2s} D_2 + \delta'_{2s} u.$$

In the second case ($\sigma < 1$), by proceeding in an analogous manner, denoting by $M_\nu(k^2 \operatorname{cn}^2 u)$ certain polynomials in k^2 and $k^2 \operatorname{cn}^2 u$ and denoting by γ_ν and γ'_ν certain polynomials in k^2, we state that we have

$$C_{2s+1} = \int_0^{\omega} (k^2 - \sin^2 \omega)^s d\omega$$

$$= \sin \omega \cos \omega\, M_{2s+1}(k^2 - \sin^2 \omega) + \gamma_{2s+1} \int_0^{\omega} d\omega$$

$$= k \operatorname{sn} u \operatorname{dn} u\, M_{2s+1}(k^2 \operatorname{cn}^2 u) + \gamma_{2s+1} C_1,$$

$$C_{2s} = \int_0^{\omega} (k^2 - \sin^2 \omega)^{\frac{2s-1}{2}} d\omega = \frac{1}{2s-1} \sin \omega \cos \omega (k^2 - \sin^2 \omega)^{\frac{2s-3}{2}}$$

$$+ \frac{2s-2}{2s-1}(2k^2-1) C_{2s-2} - \frac{2s-3}{2s-1} k^2 (2k^2-1) C_{2s-4}$$

$$= k \operatorname{sn} u \operatorname{dn} u\, k \operatorname{cn} u\, M_{2s}(k^2 \operatorname{cn}^2 u) + \gamma_{2s} C_2 + \gamma'_{2s} u$$

To form the expressions of the integrals D_1, D_2, C_1, C_2, it is convenient to introduce the notations /41

$$X(u) = \arcsin(\operatorname{sn} u) - \frac{\pi}{2K} u,$$

$$Y(u) = \arcsin(k \operatorname{sn} u),$$

$$Z(u) = \frac{d}{du} \log \vartheta_0 \left(\frac{u}{2K} \middle/ \frac{K' \sqrt{-1}}{K} \right) \qquad (7)$$

$$= -\frac{\pi}{2KK'} u + \frac{d}{du} \log \vartheta_2 \left(\frac{u \sqrt{-1}}{2K'} \middle/ \frac{K \sqrt{-1}}{K'} \right).$$

Here, ϑ_0 and ϑ_2 are functions well known from the theory of elliptic functions. The functions X(u) and Y(u) have the period 4K while the function Z(u) has the period 2K.

Using these notations, we obviously have

$$D_1 = X(u) + \frac{\pi}{2K}u.$$

$$C_1 = Y(u),$$

and, in accordance with the theory of elliptic functions (formulas by Schwarz, pp.52, 62, 63),

$$D_2 = Z(u) + \frac{E}{K}u,$$

$$C_2 = Z(u) + \left(\frac{E}{K} + k^2 - 1\right)u.$$

(Here, E is the complete integral of the second kind, corresponding to the modulus k.)

Let us now return to eqs.(3) and (4).

The mean value [Q] of the function Q(u) can be expanded in powers of μ or $\sqrt{\mu}$. According to the above statements, the coefficients of the various powers of μ or $\sqrt{\mu}$ are expressions which, in the first case ($\sigma > 1$), have the form

$$\alpha_1 + \beta_1 \frac{E}{K} + \gamma_1 \frac{\pi}{2K};$$

and, in the second case ($\sigma < 1$), have the form /42

$$\alpha_2 + \beta_2 \frac{E}{K}.$$

The quantities α_1, β_1, γ_1 are polynomials in k and k^{-1}, while α_2 and β_2 are polynomials in k.

The function [Q] $\cdot$ $\widetilde{Q}(u)$ can be expanded in powers of μ or $\sqrt{\mu}$. The various terms of the series, in the first case ($\sigma > 1$), have the form

$$a_1 X(u) + b_1 Z(u) + \operatorname{sn} u \operatorname{cn} u \, L(\operatorname{dn} u);$$

and, in the second case ($\sigma < 1$),

$$a_2 Y(u) + b_2 Z(u) + k \operatorname{sn} u \operatorname{dn} u \, M(k \operatorname{cn} u).$$

Here, L and M are polynomials in dn u resp. k cn u; a_1, b_1, a_2, b_2 are constants; a_1, b_1, and the coefficients of L are polynomials in k and k^{-1}; a_2, b_2 and the coefficients of M are polynomials with respect to k.

Let us finally return to the general equation (6). Consider a function $\Pi(u)$ expandable in powers of μ or $\sqrt{\mu}$, where the various terms are polynomials;

in the first case ($\sigma > 1$) with respect to the quantities

$$\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u, X(u), Z(u);\ k, k^{-1}, \frac{\pi}{2K}, \frac{E}{K}; \tag{8}$$

and, in the second case ($\sigma < 1$), with respect to the quantities

$$\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u, Y(u), Z(u);\ k, \frac{E}{K}. \tag{8'}$$

Let us substitute, in this function $\Pi(u)$, the quantity u by the variable v. In view of eq.(6) and of the character of the function $\mathfrak{Q}(v)$ established above, it is obvious that the function $\Pi(u)$, considered as a function of v, can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms being polynomials; in the first case, with respect to the quantities

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$$\operatorname{sn} v, \operatorname{cn} v, \operatorname{dn} v, X(v), Z(v);\ k, k^{-1}, \frac{\pi}{2K}, \frac{E}{K}; \tag{9}$$

and, in the second case, with respect to the quantities

$$\operatorname{sn} v, \operatorname{cn} v, \operatorname{dn} v, Y(v), Z(v);\ k, \frac{E}{K}. \tag{9'}$$

The functions (9) and (9') are periodic in v, with the period 2K or 4K. Consequently, it will be possible to expand these functions as well as the various terms of $\Pi(u)$ considered as a function of v, in Fourier series in multiples of the argument

$$w = \frac{\pi}{2K} v = \frac{\pi}{2K} \mu^{2+\frac{\epsilon}{2}} \frac{N' + N''}{[Q]} t + \gamma. \tag{10}$$

However, these Fourier series converge too slowly as soon as k becomes close to unity. It is preferable to retain the various terms of the expansion of the function Π, expressed as polynomials of the functions (9) resp. (9') and to calculate these functions (9) resp. (9') over the intermediary of the functions ϑ.

The elliptic functions sn v, cn v, dn v are expressed, over the intermediary of the functions ϑ, by the following formulas:

$$\operatorname{sn} v = 2K \frac{\vartheta_0(0/\tau)}{\vartheta_1'(0/\tau)} \frac{\vartheta_1\left(\frac{v}{2K}/\tau\right)}{\vartheta_0\left(\frac{v}{2K}/\tau\right)} =$$

$$= \frac{2K'}{\sqrt{-1}} \frac{\vartheta_2\left(0/-\frac{1}{\tau}\right)}{\vartheta_1'\left(0/-\frac{1}{\tau}\right)} \frac{\vartheta_1\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)}{\vartheta_2\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)}, \tag{11}$$

$$\operatorname{cn} v = \frac{\vartheta_0(0/\tau)}{\vartheta_2(0/\tau)} \frac{\vartheta_2\left(\frac{v}{2K}/\tau\right)}{\vartheta_0\left(\frac{v}{2K}/\tau\right)} = \frac{\vartheta_2\left(0/-\frac{1}{\tau}\right)\vartheta_0\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)}{\vartheta_0\left(0/-\frac{1}{\tau}\right)\vartheta_2\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)},$$

$$\operatorname{dn} v = \frac{\vartheta_0(0/\tau)}{\vartheta_3(0/\tau)} \frac{\vartheta_3\left(\frac{v}{2K}/\tau\right)}{\vartheta_0\left(\frac{v}{2K}/\tau\right)} = \frac{\vartheta_2\left(0/-\frac{1}{\tau}\right)\vartheta_3\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)}{\vartheta_3\left(0/-\frac{1}{\tau}\right)\vartheta_2\left(\frac{v\sqrt{-1}}{2K'}\Big/-\frac{1}{\tau}\right)}.$$

For abbreviation, we have put there /44

$$\tau = \frac{K'}{K}\sqrt{-1}.$$

These formulas must be complemented by eqs.(7), writing there only v instead of u.

Below, we give the expressions for the functions ϑ:

$$\begin{aligned}
\vartheta_0(x/\tau) &= 1 - 2q\cos 2x\pi + 2q^4\cos 4x\pi - 2q^9\cos 6x\pi + \cdots,\\
\vartheta_1(x/\tau) &= 2q^{1/4}\sin x\pi - 2q^{9/4}\sin 3x\pi + 2q^{25/4}\sin 5x\pi - \cdots,\\
\vartheta_2(x/\tau) &= 2q^{1/4}\cos x\pi + 2q^{9/4}\cos 3x\pi + 2q^{25/4}\cos 5x\pi + \cdots,\\
\vartheta_3(x/\tau) &= 1 + 2q\cos 2x\pi + 2q^4\cos 4x\pi + 2q^9\cos 6x\pi + \cdots;
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\vartheta_0\left(x'\sqrt{-1}/-\frac{1}{\tau}\right) =\\
&\qquad = 1 - q'(e^{2x'\pi} + e^{-2x'\pi}) + q'^4(e^{4x'\pi} + e^{-4x'\pi}) - \cdots,\\
&\frac{1}{\sqrt{-1}}\vartheta_1\left(x'\sqrt{-1}/-\frac{1}{\tau}\right) =\\
&\qquad = q'^{1/4}(e^{x'\pi} - e^{-x'\pi}) - q'^{9/4}(e^{3x'\pi} - e^{-3x'\pi}) + \cdots,\\
&\vartheta_2\left(x'\sqrt{-1}/-\frac{1}{\tau}\right) =\\
&\qquad = q'^{1/4}(e^{x'\pi} + e^{-x'\pi}) + q'^{9/4}(e^{3x'\pi} + e^{-3x'\pi}) + \cdots,\\
&\vartheta_3\left(x'\sqrt{-1}/-\frac{1}{\tau}\right) =\\
&\qquad = 1 + q'(e^{2x'\pi} + e^{-2x'\pi}) + q'^4(e^{4x'\pi} + e^{-4x'\pi}) + \cdots,
\end{aligned} \tag{13}$$

with the notations

$$q = e^{\tau\pi\sqrt{-1}} = e^{-\frac{K'}{K}\pi}, \quad q' = e^{-\frac{\pi}{\tau}\sqrt{-1}} = e^{-\frac{K}{K'}\pi}.$$

By means of all these formulas, the function (9) resp. (9') is always readily calculated. If $k^2 < \frac{1}{2}$, we have $K < K'$ and $q < 0.04321$. It then is useful to apply the trigonometric series (12). If, conversely, $k^2 > \frac{1}{2}$, then we have $K > K'$ and $q' < 0.04321$. In this case, the exponential series (13) are to be preferred. Evidentally, it is sufficient to consider there only the values of v in the domain /45

$$0 \leq v < 2K.$$

This will yield

$$1 < e^{x'\pi} = e^{\frac{v\pi}{2K'}} < q'^{-1}.$$

According to the above statements, it is obvious that the functions periodic in v and having a period 4K, which enter as coefficients of the various powers of μ or $\sqrt{\mu}$ into the expansions of the functions $\Pi(u)$, remain finite and well-determined even in the case in which the modulus k infinitely approaches unity. By putting k = 1, we have the limiting case which separates the first case from the second case. This will yield

$$K = \infty, \quad K' = \frac{\pi}{2}, \quad q' = 0;$$

$$\operatorname{sn} v = Z(v) = \frac{e^v - e^{-v}}{e^v + e^{-v}},$$

$$\operatorname{cn} v = \operatorname{dn} v = \frac{2}{e^v + e^{-v}},$$

$$X(v) = Y(v) = \arcsin\left(\frac{e^v - e^{-v}}{e^v + e^{-v}}\right).$$

The function Π no longer is periodic but asymptotically approaches a value constant for $v = \infty$.

We can then replace Π by any of the following functions:

$$u, \quad \varrho, \quad \sqrt{\varrho^2 - \varkappa}, \quad \xi = \varrho \cos \omega, \quad \eta = \varrho \sin \omega.$$

Thus, it becomes possible to calculate the solution of the system (1) of Section 10 for the typical cases (ϵ = 2, 3, 4, 5) considered in the present Section.

In the first case ($\sigma > 1$), the argument ω may increase indefinitely and possesses the mean motion $\frac{dw}{dt}$ which is at least of the order of μ^3 [see eq.(10)]. /46

In the second case ($\sigma < 1$) libration is present and the argument ω remains enclosed between two limits. Its mean motion is zero.

In the two cases, the period $2\pi : \frac{dw}{dt}$ of the solution of the investigated canonical system is extremely long and at least of the order of μ^{-3}. The period tends toward infinite as soon as the limiting case is approached where $k = 1$.

In the second case (case of libration), we can also put $k = 0$. This will yield

$$K = \frac{\pi}{2}, \quad K' = \infty, \quad q = 0;$$

$$\operatorname{sn} v \equiv \sin v, \quad \operatorname{cn} v \equiv \cos v, \quad \operatorname{dn} v \equiv 1,$$

$$X(v) \equiv Y(v) \equiv Z(v) \equiv 0.$$

Because of the nature of the function $\tilde{Q}(v)$, we also have

$$\tilde{Q}(v) \equiv 0, \quad u \equiv v.$$

In addition, in this entirely particular case, we have

$$\sin \omega = k \sin \omega' = 0.$$

The point (ξ, η) coincides with one of the libration points P_5 (see Section 9).

This leaves a rather exceptional case to be treated, by assuming that the integration constants $\varkappa$ and h take values such that none of the ratios

$$\frac{\Delta_0}{\Delta_4}, \quad \frac{\Delta_2}{\Delta_4}$$

will be large while all the ratios /47

$$\frac{\Delta_6}{\Delta_4}, \quad \frac{\Delta_8}{\Delta_4}, \ldots$$

will be small. Then, the discriminant D can be given the form

$$D = \mu^6 (\cos^2 2\omega + \alpha \cos 2\omega + \beta) g,$$

where α and β are two finite arbitrary constants which can replace $\varkappa$ and h, while g is a function of ω differing from zero for $\mu = 0$ and expandable in powers of μ.

In this case, one or the other of the functions ρ and $\sqrt{\rho^2 - \varkappa}$ is of the order of $\sqrt{\mu}$. We now are in the vicinity of the contour curves represented in Figs.1 and 3 of Section 9.

It is then expedient to replace ω by an auxiliary variable u, defined by the relation

$$u = \int_0^\omega \frac{d\omega}{\sqrt{\cos^2 2\omega + \alpha \cos 2\omega + \beta}}.$$

This again leads us back to elliptic functions. Also in this case, it is possible to form the solution of eqs.(1) of Section 10. However, a complete analytical discussion would lead too far here.

Similarly, we will not discuss the more special case in which the three first coefficients Δ_0, Δ_2, Δ_4 are either comparable in magnitude with Δ_6 or else are small with respect to Δ_6. In this case, if it can be realized at all, the three parameters x_1^*, $\varkappa$, and h are quite close to certain special values that satisfy the equations

$$\Delta_0 = \Delta_2 = \Delta_4 = 0.$$

Section 13.

We will continue the integration of the system of secular inequalities, by again making use of the conditions given in Section 11.

Equation (1) of Section 11 can be written as

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$$\frac{d\omega}{dt} = \frac{(N' + N'')\mu^{2+\frac{\varepsilon}{2}}}{P(\omega)} = R(\omega). \tag{1}$$

The second member of this expression can be expanded in powers of μ or $\sqrt{\mu}$, and the various terms of this expansion are finite trigonometric series in cosines of multiples of the argument 2ω. The constant term of $R(\omega)$ is of the order of $\mu^{2+\frac{\varepsilon}{2}}$; the principal periodic term, namely that in cos 2ω, is at least of the order of $\mu^{3+\frac{\varepsilon}{2}}$.

This equation (1) has been used already for expressing ω - w as a periodic function of a linear argument w. The difference ω - w is of the order of μ^2 if ε = 0 (case of regular planets) and of the order of μ if ε = 1, 2, 3, 4, 5.

Let us now pass to eq.(18) of Section 8, by writing it in the form of

$$\frac{d\omega_0''}{dt} = 2\mu \frac{dH_0}{dx} = S(\omega). \tag{2}$$

The second term can be expanded in powers of μ, where the various terms are polynomials in ρ^2 and $\rho^2(\rho^2 - \varkappa)\cos 2\omega$. The character of the four first of these polynomials is obtained from eqs.(19) of Section 8, after differentiation with respect to $\varkappa$. We will replace ρ^2 by its expansion in powers of μ or $\sqrt{\mu}$ in accordance with eq.(8) of Section 10. Let us note that the periodic part of ρ^2 is of the order of μ^2 if $\epsilon = 0, 2, 3, 4, 5$ and of the order of $\mu^{3/2}$ if $\epsilon = 1$. Thus, the function $S(\omega)$ can be expanded in powers of μ or $\sqrt{\mu}$. The various terms of its expansion are finite trigonometric cosine series of multiples of 2ω. The constant term of $S(\omega)$ is of the order of μ; the principal periodic term, namely that in $\cos 2\omega$, is of the order of μ^4 if $\epsilon = 0, 2, 3, 4, 5$ and of the order of $\mu^{7/2}$ if $\epsilon = 1$.

Equation (2) can be used for determining ω''_* as a function of t. In eq.(5) of Section 11, we will then substitute $S(\omega)$ for $\Pi(\omega)$. Thus, $\frac{d\omega''}{dt}$ appears as a trigonometric series, ordered in cosines of multiples of the linear argument 2w. Let /49

$$\mu \nu''$$

be the constant term of this trigonometric series. The quantity ν'' can be expanded in powers of μ or $\sqrt{\mu}$, where the first term is ν''_0. Putting

$$w'' = \mu \nu'' t + \gamma'',$$

where γ'' is an arbitrary constant, we find after integration that the difference $\omega'' - w''$ can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms being finite trigonometric sine series of multiples of the argument 2w. The principal coefficient, namely, that of sin 2w in the trigonometric series of $\omega''_* - w''$, is of the order of μ^2 if $\epsilon = 0$, while it is of the order of μ if $\epsilon = 1$ and of the order of $\mu^{2-\frac{\epsilon}{2}}$ if $\epsilon = 2, 3, 4, 5$. The ratios of the other coefficients to the principal coefficient are at least of the order of μ.

After having expressed the differences $\omega - w$ and $\omega''_* - w''$ as periodic functions of the linear argument w, it is easy to give the complete solution of eqs.(1) of Section 3, always remaining within the assumptions of Section 11. In fact, we first have the relations (17) of Section 3 which connect the variables

$$\xi_1^*,\ \eta_1^*,\ \xi_2^*,\ \eta_2^* \tag{4}$$

to the variables

$$\xi',\ \eta',\ \xi'',\ \eta''. \tag{5}$$

Then, by means of the transformation (12) of Section 8, these latter variables are expressed as functions of the variables

$$\xi'_* = \varrho \cos(\omega - \omega''_*), \quad \eta'_* = \varrho \sin(\omega - \omega''_*),$$
$$\xi''_* = \sqrt{\varrho^2 - \varkappa}\cos\omega''_*, \quad \eta''_* = \sqrt{\varrho^2 - \varkappa}\sin\omega''_*. \tag{6}$$

Obviously, the variables (4) are of the order of $\sqrt{\mu}$. They can be expanded in odd powers of $\sqrt{\mu}$. The coefficients of $(\sqrt{\mu})^{2i+1}$ in these expansions are polynomials of the degree $2i + 1$ with respect to the variables (6). Instead of ρ and $\sqrt{\rho^2 - \varkappa}$, we now introduce their expressions as functions of ω in accordance with Section 11. Thus the variables (4), divided by $\sqrt{\mu}$, are finally expanded in powers of μ or $\sqrt{\mu}$, with the various terms being finite trigonometric series of the two arguments ω''_* and ω, which are known as functions of t. /50

Next, we will integrate eqs.(2) of Section 3. We noted above that the second of these equations simply yields

$$y_2^* = t. \tag{7}$$

In the first of these equations, we can put

$$y_1^* = nt + c + \chi, \tag{8}$$

where n is a still unknown constant, c is an arbitrary constant, and χ is a periodic function with respect to the two arguments ω''_* and ω. Since eqs.(1) and (2) are given, the equation which will yield the function χ assumes the form

$$\frac{d\chi}{d\omega''_*} S(\omega) + \frac{d\chi}{d\omega} R(\omega) = -\frac{dF^*}{dx_1^*} - n. \tag{9}$$

By writing only the principal terms, we have

$$S(\omega) = \frac{dw''}{dt} + \cdots, \quad R(\omega) = \frac{dw}{dt} + \cdots.$$

These terms are, respectively, of the order of μ and of the order of $\mu^{2+\frac{\epsilon}{2}}$. The terms which are not given here are smaller.

Let us first express the second member of eq.(9) as a function of the variables (6). In eq.(6) of Section 6, we gave the series

$$\frac{dF^*}{dx_1^*} - \frac{dF_0^*}{dx_1^*} - \mu \frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1^*} = \mu^2 \sum_{m=0}^{\infty} \mu^m G^{(m)},$$ /51

where the $G^{(m)}$ are certain polynomials of the degree $2m + 2$ with respect to the variables (5). After performing the transformation (12) of Section 8, we will have

$$\sum_{m=0}^{\infty} \mu^m G^{(m)} = \sum_{m=0}^{\infty} \mu^m G_*^{(m)},$$

where $G_*^{(m)}$ are certain polynomials of the degree $2m + 2$ with respect to the variables (6). Specifically, using the notations of Section 6, we obtain

$$\begin{aligned} G^{(0)} &= G_{0,0}^{0,0,0} + 2G_{1,0}^{0,1,0}\xi' + G_{0,0}^{0,2,0}(\varrho'^2 - \varrho''^2) \\ G_*^{(0)} &= G_{0,0}^{0,0,0} + 2G_{1,0}^{0,1,0}\xi'_* + G_{0,0}^{0,2,0}(\varrho'^2_* - \varrho''^2_*) \\ &= G_{0,0}^{0,0,0} + G_{0,0}^{0,2,0}\varkappa + 2G_{1,0}^{0,1,0}\varrho\cos(\omega - \omega''_*). \end{aligned} \tag{10}$$

In accordance with Section 11, the quantities ρ and $\sqrt{\rho^2 - \varkappa}$, which enter the expressions (6) of the variables ξ'_*, η'_*, ξ''_*, η''_* can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms being finite trigonometric cosine series of multiples of the argument 2ω. Thus, the derivative $\frac{dF^*}{dx_1{}^*}$ in the second member of eq.(9) can also be expanded in powers of μ or $\sqrt{\mu}$, with the various terms of the expansion being finite trigonometric series of the cosines of multiples of the two arguments ω''_* and ω.

In view of this, it is easy to integrate the equation of partial derivatives (9) and to express χ as a periodic function of ω''_* and ω.

Let f be any periodic function with respect to the arguments ω''_* and ω. Let us denote by $\{f\}$ the part of f which is independent of ω''_*.

Thus, eq.(9) is partitioned into the two following equations, which are completely independent of each other,

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$$\frac{d(\chi - \{\chi\})}{d\omega''_*} S(\omega) + \frac{d(\chi - \{\chi\})}{d\omega} R(\omega) = -\frac{dF^*}{dx_1{}^*} + \left\{\frac{dF^*}{dx_1{}^*}\right\}, \tag{11}$$

$$\frac{d\{\chi\}}{d\omega} R(\omega) = -\left\{\frac{dF^*}{dx_1{}^*}\right\} - n. \tag{12}$$

These equations, obviously, can be satisfied by putting, for $\chi - \{\chi\}$, $\{\chi\}$, and n, certain series in powers of μ or $\sqrt{\mu}$.

The successive terms of the expansion of $\chi - \{\chi\}$ are finite trigonometric sine series of multiples of the two arguments ω''_* and ω. The function $\chi - \{\chi\}$ is, in addition, of the order of μ. This is due to the fact that $\frac{dw''}{dt}$ is larger than $\frac{dw}{dt}$ and is of the order of μ, while the second member of eq.(11) is of the order of μ^2. [See the expression (10) of $G_*^{(0)}$.]

In the expansion of the function $\{\chi\}$, the various terms are finite trigonometric sine series of multiples of the argument 2ω.

We know that $\{\chi\}$ is of the order of μ^3 for the so-called regular planets (see Section 6). It follows from this that the functions $\{G_*^{(1)}\}$ and $\{G_*^{(2)}\}$ are polynomials in ρ^2 with constant coefficients. Incidentally, this could have been demonstrated directly.

In view of this, it is easy in general to find the order of magnitude of the function $\{\chi\}$. We know that the variable part of ρ^2, considered as a function of ω, is of the order of $\mu^{3/2}$ for $\epsilon = 1$ and of the order of μ^2 for $\epsilon = 0$, 2, 3, 4, 5. Since also the indicated form of the functions $\{G_*^{(1)}\}$ and $\{G_*^{(2)}\}$ is given, it can be concluded that the variable part of the function $\frac{dF^*}{dx_1{}^*}$ is of the order of $\mu^{3/2}$ for $\epsilon = 1$ and of the order of μ^5 for $\epsilon = 0, 2, 3, 4, 5$. In addition, the principal term of the function $R(\omega)$ is constant and of the order of $\mu^{3+\frac{\epsilon}{2}}$. From this it follows that the function $\{\chi\}$ is

of the order	μ^3,	μ^2,	μ^2,	$\mu^{3/2}$,	μ,	$\mu^{1/2}$,
for $\epsilon =$	0,	1,	2,	3,	4,	5.

Let us finally pass to the constant n. Below, we give the first terms of its expansion: /53

$$n = x_1{}^{*-3} - \mu \frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1{}^*} - \mu^2 (G^{0,0,0}_{0,0} + G^{0,2,0}_{0,0}\,\chi) - \cdots. \tag{13}$$

Section 14.

The problem will be treated here in a manner analogous to the case in Section 12.

Equation (2) resp. eq.(2') of Section 12 can be written as

$$\frac{du}{dt} = \frac{(N' + N'')\,\mu^{2+\frac{\epsilon}{2}}}{Q(u)} = T(u). \tag{1}$$

The second member of this equation can be expanded in powers of μ if $\epsilon = 2$, 4 and in powers of $\sqrt{\mu}$ if $\epsilon = 3, 5$. The successive terms of this development are polynomials; in the first case ($\sigma > 1$) with respect to k, k^{-1}, and dn u; in the second case ($\sigma < 1$) with respect to k and k cn u. The term independent of dn u resp. k cn u is of the order of $\mu^{2+\frac{\epsilon}{2}}$. The other terms are at least of the order of $\mu^{3+\frac{\epsilon}{2}}$.

We have demonstrated above that the difference u - v can be expressed as a periodic function of the linear argument v, defined by eq.(3) in Section 12. To arrive at this result, it is sufficient to replace $\Pi(u)$ in the general equation (6) of Section 12 by u. In this manner, we find

$$u-v=\sum_{s=0}^{\infty}\frac{\mu^{s+1}}{s+1!}\frac{d^s}{dv^s}(\tilde{Q}(v))^{s+1}. \tag{2}$$

Thus, the function u - v is expanded in powers of μ or $\sqrt{\mu}$ depending on whether ϵ is even or odd. The various terms of the expansion are polynomials; in the first case ($\sigma > 1$), with respect to the quantities (9) of Section 12; in the second case ($\sigma < 1$), with respect to the quantities (9') of Section 12. The difference u - v is of the order of μ^2 for $\epsilon = 2, 3, 4$ and of the order of μ for $\epsilon = 5$. This is the result of our above statements as to the order of magnitude of the function $\tilde{Q}(v)$ on p.77.

It would be possible to substitute the argument w for v by eq.(10) of Section 12. Thus, the difference $\frac{\pi}{2K}$ u - w appears as a periodic function in w, having a period of 2π. In view of the expansion of [Q], indicated on p.76, it is obvious that the velocity $\frac{dw}{dt}$ of the argument w can be expanded in powers of μ or $\sqrt{\mu}$, depending on whether ϵ is even or odd, and that the various terms of the series are polynomials; in the first case ($\sigma > 1$), with respect to the quantities /54

$$k^{-1},\quad k,\quad \frac{E}{K},\quad \frac{\pi}{2K} \tag{3}$$

and, in the second case ($\sigma < 1$), with respect to the quantities

$$k,\quad \frac{E}{K}. \tag{3'}$$

Let us now pass to eq.(18) of Section 8, written in the form of

$$\frac{d\omega_*'}{dt}=2\mu\frac{dH_*}{dx}=U(u). \tag{4}$$

We wish to express ω_*'' as a function of the linear argument v. By putting

$$V(u)=\frac{U(u)[Q]}{(N'+N'')\mu^{2+\frac{\epsilon}{2}}}$$

and by applying eq.(6) of Section 12, we obtain

$$\frac{d\omega_*''}{dv} = V(u) = V(v) + \mu \frac{dV(v)}{dv}\tilde{Q}(v)$$
$$+ \sum_{s=1}^{\infty} \frac{\mu^{s+1}}{s+1!}\frac{d^s}{dv^s}\left(\frac{dV(v)}{dv}(\tilde{Q}(v))^{s+1}\right). \tag{5}$$

It is convenient to integrate the second term in steps so as to avoid the appearance of the function $\tilde{Q}(v)$ under the integration sign. In view of /55

$$\mu\frac{d\tilde{Q}(v)}{dv} = 1 - \frac{Q(v)}{[Q]},$$

the following expression is found for the integral of the second term:

$$\mu(V(v)-[V])\tilde{Q}(v) + \int (V(v)-[V])\left(\frac{Q(v)}{[Q]}-1\right)dv.$$

Here, we denoted by [V] the mean value of the periodic function V(v).

Thus, after integration, eq.(4) will yield

$$\omega_*'' = [V]v + \int (V(v)-[V])\frac{Q(v)}{[Q]}dv + \mu(V(v)-[V])\tilde{Q}(v)$$
$$+ \sum_{s=1}^{\infty}\frac{\mu^{s+1}}{s+1!}\frac{d^{s-1}}{dv^{s-1}}\left(\frac{dV(v)}{dv}(\tilde{Q}(v))^{s+1}\right).$$

The secular part of the second member has the expression

$$\frac{[VQ]}{[Q]}v \equiv \frac{[UQ]}{[Q]}t.$$

In view of this, we can put

$$\mu\nu'' = \frac{[UQ]}{[Q]}, \quad w'' = \mu\nu''t + \gamma'', \tag{6}$$

where γ'' is an arbitrary constant. For abbreviation, we will introduce a notation by putting

$$W(u) = (V(u)-[V])\frac{Q(u)}{[Q]} = \frac{U(u)-[U]}{(N'+N'')\mu^{2+\frac{\varepsilon}{\gamma}}}Q(u). \tag{7}$$

This will finally yield the wanted expression

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$$\omega''_* - w'' = \int (W(v) - [W])\,dv + \mu\,(V(v) - [V])\,\tilde{Q}(v) + \sum_{s=1}^{\infty} \frac{\mu^{s+1}}{s+1!} \frac{d^{s-1}}{dv^{s-1}} \left(\frac{dV(v)}{dv} (\tilde{Q}(v))^{s+1} \right). \tag{8}$$

The velocity $\mu\nu''$ of the argument w'' can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms being polynomials; in the first case ($\sigma > 1$), with respect to the quantities (3); in the second case ($\sigma < 1$), with respect to the quantities (3'). For $\mu = 0$, we have $\nu'' = \nu''_0$.

The functions $(V(u) - [V])$ and $W(u)$ can be expanded in powers of μ or $\sqrt{\mu}$, with the successive terms being polynomials; in the first case ($\sigma > 1$), with respect to the quantities dn u, k^{-1}, k, $\frac{E}{K}$, $\frac{\pi}{2K}$; in the second case ($\sigma < 1$), with respect to k cn u, k, and $\frac{E}{K}$. It follows from this that the function (8) can be expanded in powers of μ or $\sqrt{\mu}$, depending on whether ϵ is even or odd, and that the various terms of this expansion are polynomials; in the first case ($\sigma > 1$), with respect to the quantities (9) of Section 12; in the second case ($\sigma < 1$), with respect to the quantities (9') of the same Section.

It is easy to give the principal term of the difference $\omega''_* - w''$ by neglecting μ with respect to unity. In fact, since the formulas (19) of Section 9 as well as the principal periodic part of ρ^2 considered as a function of u are given, it is easy to find that the function W(u), in the considered approximation, is a polynomial of the second degree with respect to dn u resp. k cn u, depending on whether the first case ($\sigma > 1$) or the second case ($\sigma < 1$) is involved. Thus, neglecting μ with respect to unity, the function $\omega''_* - w''$ is homogeneous and linear; in the first case, with respect to the functions X(v) and Z(v); in the second case, with respect to Y(v) and Z(v). In the most important case in which $\epsilon = 2$, we readily obtain

$$\omega''_* - w'' = \frac{N''}{N' + N''} X(v) + \cdots = \delta_0\,\omega''_* + \cdots, \quad \text{if}\ \sigma > 1;$$

$$\omega''_* - w'' = \frac{N''}{N' + N''} Y(v) + \cdots = \delta_0\,\omega''_* + \cdots, \quad \text{if}\ \sigma < 1.$$

In the less frequent cases in which $\epsilon = 3$, 4, or 5, it is necessary to add a term in Z(v). This second term is of the order of $\mu^{\frac{1}{2}}$ for $\epsilon = 3$, of the order of μ^0 for $\epsilon = 4$, and of the order of $\mu^{-\frac{1}{2}}$ for $\epsilon = 5$. /57

It is highly interesting that the argument ω''_* thus contains, under the conditions of Section 12, a periodic term which does not vanish with μ. This term will be denoted by

$$\delta_0 \omega''_*.$$

From the various transformations which link the variables x_1, y_1, ξ_k, η_k of Section 2 to the variables ξ'_*, η'_*, ξ''_*, η''_* of Section 8, it is easy to demonstrate that the argument $-\omega''_*$ differs from the longitude of the node θ only by small periodic quantities of the order of μ. Thus, the quantity $-\delta_0\omega''_*$ is nothing else but the most important periodic inequality in the longitude of the node of the orbit of the minor planet.

The argument ω'_* contains an analogous inequality which we will denote by $\delta_0\omega'_*$ and which is obtained by permuting N' and N'' in the above factor of $X(v)$ resp. $Y(v)$.

The corresponding inequality in the argument $\omega = \omega'_* + \omega''_*$ thus will be $X(v)$ resp. $Y(v)$ which could be predicted from the formulas (7) of Section 12.

After having expressed the differences $\frac{\pi}{2K}$ u - w and $\omega''_* - w''$ as periodic functions of the linear argument w with the period 2π, it is easy to obtain the complete solution of eqs.(1) of Section 3 under the assumptions of Section 12. We know that the variables

$$\xi_1^*, \ \eta_1^*, \ \xi_2^*, \ \eta_2^* \qquad (9)$$

of Section 3 can be expanded in odd powers of $\sqrt{\mu}$ and that the coefficients of $(\sqrt{\mu})^{2i+1}$ in these expansions are polynomials of the degree $2i + 1$ with respect to the variables

$$\xi'_* = \varrho \cos(\omega - \omega''_*), \quad \eta'_* = \varrho \sin(\omega - \omega''_*),$$
$$\xi''_* = \sqrt{\varrho^2 - \varkappa} \cos \omega''_*, \quad \eta''_* = \sqrt{\varrho^2 - \varkappa} \sin \omega''_*.$$

We also know that the functions /58

$$\varrho \cos \omega, \quad \varrho \sin \omega, \quad \sqrt{\varrho^2 - \varkappa}$$

can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms being polynomials in sn u, cn u, and dn u. Thus, the variables (9), divided by $\sqrt{\mu}$, can be expanded in powers of μ or $\sqrt{\mu}$, where the successive terms are polynomials with respect to the functions

$$\operatorname{sn} u, \quad \operatorname{cn} u, \quad \operatorname{dn} u, \quad \sin \omega''_*, \quad \cos \omega''_*. \qquad (10)$$

It is necessary to integrate also the first of the equations of the system (2) of Section 3. As in the preceding Section, we will put

$$y_1^* = nt + c + \chi \tag{11}$$

where n is a still unknown constant, c is an arbitrary constant, and χ is an unknown function periodic with respect to the two arguments ω_*'' and u. Since eqs.(1) and (4) are given, the function χ must satisfy the equation of partial derivatives

$$\frac{d\chi}{d\omega_*''} U(u) + \frac{d\chi}{du} T(u) = -\frac{dF^*}{dx_1^*} - n. \tag{12}$$

The derivative in the second member can be expanded in powers of μ or $\sqrt{\mu}$, where the various terms are polynomials with respect to the functions (10). We can expand these terms in finite trigonometric series, arranged in accordance with multiples of the argument ω_*''. Let us denote by $\{f\}$ the mean value of any function $f(\omega_*'')$, periodic with respect to ω_*''. In view of this, eq.(12) is partitioned into two expressions:

$$\frac{d(\chi-\{\chi\})}{d\omega_*''} U(u) + \frac{d(\chi-\{\chi\})}{du} T(u) = -\frac{dF^*}{dx_1^*} + \left\{\frac{dF^*}{dx_1^*}\right\}, \tag{13}$$

$$\frac{d\{\chi\}}{du} T(u) = -\left\{\frac{dF^*}{dx_1^*}\right\} - n. \tag{14}$$

We know the nature of the functions U(u) and T(u). Their principal terms /59 are constant and of the order of μ resp. $\mu^{2+\frac{\epsilon}{2}}$. We also know, in accordance with the research reported in Section 13, that the second member of eq.(13) is of the order of μ^2. Obviously, we can satisfy eq.(13) by a function $\chi - \{\chi\}$ of the order of μ which can be expanded in powers of μ or $\sqrt{\mu}$, with the various terms of the expansion being polynomials with respect to the functions (10). Thus, the function $\chi - \{\chi\}$, divided by μ, is of the same type as the functions (9) divided by $\sqrt{\mu}$. It would be very easy to write down the first term of the considered expansion of the function $\chi - \{\chi\}$.

We can then pass to eq.(14). In accordance with the data in Section 13, we first have

$$-\left\{\frac{dF^*}{dx_1^*}\right\} = -\frac{dF_0^*}{dx_1^*} - \mu \frac{dF_{0,0,0,0}^{1,0,0,0}}{dx_1^*} - \mu^2 \{G_*^{(0)}\}$$
$$- \mu^3 \{G_*^{(1)}\} - \mu^4 \{G_*^{(2)}\} - \mu^5 \{G_*^{(3)}\} - \cdots.$$

The three first terms are constant. The terms in μ^3 and μ^4 are polynomials in ρ^2, with constant coefficients. The following terms are polynomials in ρ^2 and cos 2ω. By introducing for ρ^2 and cos 2ω their expressions as functions of u, the second member finally will be expanded in powers of μ or $\sqrt{\mu}$, with the various terms [as for the function T(u)] being polynomials in dn u in the first case ($\sigma > 1$) and polynomials in k cn u in the second case ($\sigma < 1$). The variable part of ρ^2 is of the order of $\mu^{\epsilon/2}$ if $\epsilon = 2, 3, 4$ and of the order of μ^2 if $\epsilon = 5$. It follows from this that the variable part of the second member of eq.(14) is of the order of $\mu^{3+\frac{\epsilon}{2}}$ if $\epsilon = 2, 3, 4$ and of the order of μ^5 if $\epsilon = 5$. We can divide by $T(u)$ which is approximately constant and of the order of $\mu^{2+\frac{\epsilon}{2}}$ and then select n such that the mean value of the quotient, considered as being a function of u, will vanish. Thus, the periodic function $\{\chi\}$ is obtained after quadrature, approximately like the function $\tilde{Q}(u)$ of Section 12. The function $\{\chi\}$ can be expanded in powers of μ or $\sqrt{\mu}$, depending on whether ϵ is even or odd. The terms of the expansion, in the first case ($\sigma > 1$), have the form

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$$a' X(u) + b' Z(u) + \operatorname{sn} u \operatorname{cn} u L'(\operatorname{dn} u)$$

and, in the second case,

$$a'' Y(u) + b'' Z(u) + k \operatorname{sn} u \operatorname{dn} u L''(k \operatorname{cn} u).$$

where L' and L'' are polynomials in dn u resp. k cn u. The quantities a', b', a'', b'' are constants. Here, a', b' and the coefficients of L' are polynomials with respect to the quantities (3); a'', b'' and the coefficients of L'' are polynomials with respect to the quantities (3'). The function $\{\chi\}$ obviously is of the order of μ if $\epsilon = 2, 3, 4$ and of the order of $\mu^{1/2}$ if $\epsilon = 5$.

The constant n, which is known as the mean absolute motion, can be expanded in powers of μ if $\epsilon = 2, 4$ or in powers of $\sqrt{\mu}$ if $\epsilon = 3, 5$. The various terms are polynomials; in the first case ($\sigma > 1$), with respect to the quantities (3); in the second case ($\sigma < 1$), with respect to the quantities (3'). The first terms of the expansion of n are given in eq.(13) of Section 13. The quantities (3) resp. (3') appear only in the following terms, not written here.

Section 15.

Now, all integrations of the problem have been performed. The integration constants are primarily

$$x_1^*, \ \varkappa, \ h,$$

which correspond to the three first integrals of the problem and secondly

$$\gamma, \ \gamma'', \ c,$$

which have been introduced after the three quadratures and which enter only the arguments w, w'', and nt + c.

We still have to indicate the general form which can be given to the variables x_i, y_i, ξ_k, η_k of the system (3) of Section 2. These variables are linked to the variables x_i^*, y_i^*, ξ_k^*, η_k^*, discussed in the preceding Section, over the transformations (12) of Section 3. We mentioned above that the differences

$x_1 - x_1^*$, $y_1 - y_1^*$, $\xi_k - \xi_k^*$, $\eta_k - \eta_k^*$ can be expanded in powers of μ, ξ_k^*, η_k^* and in multiples of the arguments y_1^* and t. In these expansions, we introduce the already defined expressions for $y_1^* - (nt + c)$, ξ_k^*, η_k^* as functions of the two /61 arguments ω_*'' and ω under the conditions of Sections 11 and 13, and as functions of the two arguments ω_*'' and $\frac{\pi}{2K}$ u under the conditions of Sections 12 and 14. Thus, the variables

$$x_1, \quad y_1 - (nt + c), \quad \xi_k, \quad \eta_k \tag{1}$$

are expanded in powers of $\sqrt{\mu}$. The various terms of the expansions are trigonometric series under the conditions of Sections 11 and 13, with respect to the arguments

$$t, \quad nt + c, \quad \omega_*'', \quad \omega,$$

and, under the conditions of Sections 12 and 14, with respect to the arguments

$$t, \quad nt + c, \quad \omega_*'', \quad \frac{\pi}{2K} u.$$

It is possible to replace $\omega_*'' - w''$ and $\omega - w$ resp. $\frac{\pi}{2K}$ u − w by their expressions as functions of the argument w. Thus, the variables (1) are expressed by series whose terms are arranged in powers of $\sqrt{\mu}$ and in multiples of the four arguments

$$t, \quad nt + c, \quad w'' + \delta_0 \omega_*'', \quad w.$$

In the third argument, the inequality $\delta_0 \omega_*''$ which is not small with respect to μ, must be retained. In the cases of Section 11, this inequality appears only if the exponent ϵ is 4 or 5. In the cases of Section 12, the "elementary" inequality in question will continue to exist.

In this Section, we have assumed up to now that the elliptic functions, introduced in Sections 12 and 14, are expanded in Fourier series in multiples of the argument $\frac{\pi}{2K}$ u resp. w. However, if the modulus k is close to unity, /62 these Fourier series converge too slowly. It then becomes necessary to retain the elliptic functions as quotients of the functions ϑ. Then, in the expansion of the variables (1) in powers of $\sqrt{\mu}$, the various terms are trigonometric series of the two arguments nt + c and t, with coefficients that are polynomials with respect to the functions

$$\operatorname{sn} u, \quad \operatorname{cn} u, \quad \operatorname{dn} u,$$
$$X(u) \text{ resp. } Y(u), \quad Z(u)$$
$$\sin \omega_*'', \quad \cos \omega_*''$$

or, if preferred, with respect to the functions

$$\operatorname{sn} v, \quad \operatorname{cn} v, \quad \operatorname{dn} v,$$
$$X(v) \text{ resp. } Y(v), \quad Z(v)$$
$$\sin(w'' + \delta_\circ \omega''_\circ), \ \cos(w'' + \delta_\circ \omega''_\circ).$$

which include only the linear arguments v and w".

RESEARCH ON THE MOTION OF MINOR PLANETS ***/1

Part III

H.v.Zeipel*

In Parts I and II of this research**, we discussed the principles of a general theory of "ordinary" minor planets for which the mean motion is not approximately commensurable with the mean motion of Jupiter. In this Part III, we will suppose that the ratio of the mean motion of the minor planet to that of Jupiter differs from a rational number

$$\frac{p+q}{p} \quad \left(\begin{matrix} q = 1, 2, 3 \ldots \\ p \text{ and } q \text{ first relatives} \end{matrix}\right)$$

by a quantity comparable in magnitude with the square root of the mass μ of Jupiter. Such a planet is known as a "characteristic" planet of the type $\frac{p+q}{p}$.

We will everywhere retain the canonical form of the equations. Integration is always possible from the formal viewpoint by means of semiconvergent series, assuming that the eccentricities and the inclination are small quantities. In the expressions of the coordinates and their velocities, the time appears /2 only, and in a linear manner, in the arguments of elliptical or trigonometric functions which remain finite for all real values of the arguments.

Let us briefly indicate the procedure used here.

We start from the canonical system (3) with four degrees of freedom, given in Section 2 of Part I. Among the canonical variables

$$x_k,\ y_k,\ \xi_k,\ \eta_k \qquad (k = 1, 2)$$

defined there, y_1 and $y_2 = t$ denote the mean longitudes of the asteroid and of Jupiter; ξ_1 and η_1 are of the order of the eccentricity while ξ_2 and η_2 are of the order of the inclination. We will replace these variables, in Section 16, by the following new canonical variables

$$\dot{x}_k,\ \dot{y}_k,\ \dot{\xi}_k,\ \dot{\eta}_k$$

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** See Vol.11, Nos.1 and 7 in K. Vet. Akad. Arkiv för matematik, astronomi och fysik.

*** Vol.12, No.9

which differ from the old variables only by quantities of the order of the mass μ. The characteristic function $\dot{F}$ of the new canonical system includes the two arguments $\dot{y}_1$ and $\dot{y}_2 = t$ only in the combination $p\dot{y} = p\dot{y}_1 - (p + q)\dot{y}_2$. From this it follows that the new system can be reduced to three degrees of freedom with the variables $\dot{x}_1$, $\dot{y}_1$, $\dot{\xi}_k$, $\dot{\eta}_k$. This reduction of the problem is always possible no matter what the ratio of the two mean motions might be. In addition, the reduction remains applicable no matter what the values of the eccentricities and of the inclination become. It is only necessary to assume that the two orbits do not intersect at all.

To reduce the problem further, we will limit the calculation to minor planets known as characteristic planets. In Section 17, we will introduce new canonical variables

$$x_1^*,\ y_1^*,\ \xi_k^*,\ \eta_k^* \qquad (k = 1, 2),$$

such that the differences $\dot{x}_1 - x_1^*$, $\dot{y}_1 - y_1^*$ will be of the order of q + 1, whereas the differences $\dot{\xi}_k - \xi_k^*$, $\dot{\eta}_k - \eta_k^*$ will be of the order q with respect to the small quantities ξ_k^*, η_k^*, and $\sqrt{\mu}$. The characteristic function F^* of the new canonical system is independent of the argument y_1^*. From this it follows that x_1^* is a constant and that the new system is composed of a canonical system with two degrees of freedom between the variables ξ_k^* and η_k^*, and of an equation expressing the derivative of y_1^* as a function of the variables ξ_k^* and η_k^*. The variables ξ_k^*, η_k^*, and y_1^* include all so-called secular inequalities. /3

In Section 18, we will investigate the analytical form of the characteristic function F^* in more detail. In the same Section we also will give detailed expressions for the principal terms of its expansion.

The next Sections are concerned with the integration of the equations of secular variations. We first obtain a particular solution in which ξ_k^* and η_k^* assume constant values $\xi_1^* = \bar{\xi}$, $\eta_1^* = \xi_2^* = \eta_2^* = 0$ for which the function F^* is stationary. In the general solution, the unknowns ξ_k^* and η_k^* execute small oscillations about these constant values. The unknowns ξ_k^*, η_k^* as well as the oscillating part of the argument y_1^* may ordinarily be expanded in trigonometric series of the two arguments w' and w'', linear with respect to time and having velocities of the order of μ. If $q \geq 2$, the coefficients of these expansions are rational with respect to the moduli of eccentricity and inclination (denoted by ϵ' and ϵ'' and introduced as integration constants) as well as with respect to the eccentricity e' of the orbit of Jupiter and the square root of μ. If q = 1, the mentioned coefficients are polynomials with respect to ϵ', ϵ'', e' and $\sqrt{\mu}$.

In the mentioned expansions, certain integration divisors appear. If one of these divisors becomes too small, the series used become illusory. In this case, the planet will be known as "singular". In the opposite case, the planet is designated as "regular". The singular planets of the type $\frac{p + 1}{p}$ have their mean motion in the vicinity of certain well-defined values located symmetrically

on either side of the value $\frac{p+1}{p}$. The singular planets of the type $\frac{p+2}{p}$ have a mean motion greater than the value $\frac{p+2}{p}$; incidentally, for these singular planets, the eccentricity or inclination is necessarily rather large (comparable to $\mu^{1/4}$). For singular planets of the types $\frac{p+q}{p}$ where $q \geq 3$, /4
the eccentricity and inclination are necessarily small (comparable to $\mu^{1/2}$).

In this Part III, we are concerned exclusively with "regular" planets. Despite the fact that our goal has been mainly to give a qualitative and analytical theory, we have developed a relatively large number of detailed formulas so as to make our work useful from the viewpoint of numerical applications.

The theory of secular inequalities of characteristic and singular planets can be developed more or less like the corresponding theory of ordinary and singular planets, discussed in Part II of this research.

Section 16.

We have thrown the equations of motion of a minor planet into the form (3) of Section 2. This represents a canonical system with four degrees of freedom. The investigated system returns to the general type of the equations (1) of Section 1. In the actual case, we have

$$\nu_1 = \nu_2 = 0,$$

$$h(x_1, x_2) = \frac{1}{2x_1^2} - x_2,$$

$$n_1 = -\frac{dh}{dx_1} = x_1^{-3}, \quad n_2 = -\frac{dh}{dx_2} = +1.$$

We will apply the reduction method of Section 1 by assuming that n_1 and n_2 are about at a commensurable and simple ratio. Let us consider two positive whole numbers p and q which are not too large and have no common factor. We will assume that

$$n_1 - \frac{p+q}{p}$$

is a small quantity of the order of $\sqrt{\mu}$ or smaller. Then, in the application of the method of Section 1, the small divisors will be three, i.e.,

$$p n_1 - (p+q) n_2, \ \nu_1, \ \nu_2.$$

We must start from the equation of partial derivatives /5

$$F\left(\frac{dS}{dy_k}, y_k; \frac{dS}{d\eta_k}, \eta_k\right) = F\left(x_k, \frac{dS}{dx_k}; \xi_k, \frac{dS}{d\xi_k}\right), \tag{1}$$

where $F(x_k, y_k; \xi_k, \eta_k)$ is a characteristic function of the system (3) of Section 2 while $F(x_k, y_k; \xi_k, \eta_k)$ is a new function which must be determined at the same time as the function $S(x_k, y_k; \xi_k, \eta_k)$.

Let us assume that the two functions $\dot{F}$ and S are formed. Then we must substitute in eqs.(3) of Section 2 the variables x_k; y_k; ξ_k, η_k by the new variables $\dot{x}_k$, $\dot{y}_k$; ξ_k, η_k defined by the equations

$$
\begin{aligned}
x_k &= \frac{dS(\dot{x}_k, y_k; \dot{\xi}_k, \eta_k)}{dy_k}, \quad & \dot{y}_k &= \frac{dS(\dot{x}_k, y_k; \dot{\xi}_k, \eta_k)}{d\dot{x}_k}, \\
\xi_k &= \frac{dS(\dot{x}_k, y_k; \dot{\xi}_k, \eta_k)}{d\eta_k}, \quad & \dot{\eta}_k &= \frac{dS(\dot{x}_k, y_k; \dot{\xi}_k, \eta_k)}{d\dot{\xi}_k}.
\end{aligned}
\tag{2}
$$

Then, we find the relation

$$F(x_k, y_k; \xi_k, \eta_k) = \dot{F}(\dot{x}_k, \dot{y}_k; \dot{\xi}_k, \dot{\eta}_k)$$

as well as the new canonical system

$$
\begin{aligned}
\frac{d\dot{x}_k}{dt} &= \frac{d\dot{F}}{d\dot{y}_k}, \quad & \frac{d\dot{y}_k}{dt} &= -\frac{d\dot{F}}{d\dot{x}_k}, \\
\frac{d\dot{\xi}_k}{dt} &= \frac{d\dot{F}}{d\dot{\eta}_k}, \quad & \frac{d\dot{\eta}_k}{dt} &= -\frac{d\dot{F}}{d\dot{\xi}_k}.
\end{aligned}
\qquad (k = 1, 2)
\tag{3}
$$

Let us demonstrate now how the functions $\dot{F}$ and S must be formed. For this purpose, it is necessary to introduce in eq.(1), the series

$$
\begin{aligned}
\dot{F} &= \dot{F}_0 + \mu \dot{F}_1 + \mu^2 \dot{F}_2 + \cdots, \\
S &= S_0 + \mu S_1 + \mu^2 S_2 + \cdots
\end{aligned}
$$

and to equate, in the expansions of the two members of eq.(1), the coefficients of μ^0, of μ, etc. By putting first

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$$\dot{F}_0 = F_0 = \frac{1}{2x_1^2} - x_2,$$

$$S_0 = \sum_{k=1}^{2} x_k y_k + \sum_{k=1}^{2} \xi_k \eta_k,$$

eq.(1) will be satisfied for $\mu = 0$.

By equating the coefficients of μ in the two members of eq.(1), we find the relation

$$\sum_{k=1}^{2} n_k \frac{dS_1}{dy_k} = F_1 - \dot{F}. \qquad (4)$$

As done frequently, we will again put

$$\xi_k = \varrho_k \cos \omega_k, \qquad \eta_k = \varrho_k \sin \omega_k.$$

Since the expression of F_1 is given by eq.(5) of Section 2, we must set

$$\dot{F}_1 = \sum F^{1,\bar{m},m_1,m_2}_{\iota p, -\iota(p+q), j_1, j_2} e^{\prime \bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \cos(\iota p y + j_1 \omega_1 + j_2 \omega_2) \qquad (5)$$

with the notation

$$py = py_1 - (p+q) y_2.$$

In the sum (5), the indices ι, j_1, j_2, $\bar{m}$, m_1, m_2 all take integral values which satisfy the conditions

$$|j_1| \leq\leq m_1,^1 \qquad |j_2| \leq\leq m_2 = \text{even},$$
$$|\iota q + j_1 + j_2| \leq\leq \bar{m}. \qquad *(6)$$

From this it follows that

$$|\iota q| \leq\leq \bar{m} + m_1 + m_2.$$

After this selection of $\dot{F}_1$, eq.(4) will yield the function S_1 without small divisors. We then find /7

$$S_1 = \sum{}' \frac{F^{1,\bar{m},m_1,m_2}_{i_1,i_2,j_1,j_2}}{i_1 n_1 + i_2 n_2} e^{\prime \bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \sin(i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2).$$

In the sum Σ', the indices must not only satisfy the conditions (6) of Section 2 but also the inequality

$$i_1(p+q) + i_2 p \neq 0.$$

Let us also equate the coefficients of μ^2 in the two members of eq.(1). This will yield the equation

* The notation $a \leq\leq b$ is to indicate that $b - a$ is a nonnegative even integer.

$$\sum_{k=1}^{2} n_k \frac{dS_2}{dy_k} = \bar{F}_2 - \dot{F}_2, \tag{7}$$

by putting, for abbreviation,

$$\bar{F}_2 = \frac{3}{2} x_1^{-4} \left(\frac{dS_1}{dy_1}\right)^2 + \frac{dF_1}{dx_1}\frac{dS_1}{dy_1} - \frac{d\dot{F}_1}{dy_1}\frac{dS_1}{dx_1} + \sum_{k=1}^{2}\left(\frac{dF_1}{d\xi_k}\frac{dS_1}{d\eta_k} - \frac{d\dot{F}_1}{d\eta_k}\frac{dS_1}{d\xi_k}\right). \tag{8}$$

In view of the form of the functions F_1, $\dot{F}_1$, and S_1 as well as of the conditions (6) of Section 2, it is easy to demonstrate that $\bar{F}_2$ will have a form analogous to that of $\bar{F}_1$ [see eq.(5) in Section 2]. Only, to obtain the conditions of the indices in $\bar{F}_2$, it is necessary to write $\bar{m} + 2$ instead of $\bar{m}$ in the conditions (6) of Section 2. If we do this, then $\dot{F}_2$ will be the sum of the terms of $\bar{F}_2$ where

$$i_1 = \iota p, \qquad i_2 = -\iota(p+q), \qquad (\iota = 0, \pm 1, \pm 2, \dots).$$

After this selection of $\dot{F}_2$, the function S_2 is obtained readily and without small divisors, after integrating eq.(7).

We can continue in this manner and thus form successively the functions $\bar{F}_i$, $\dot{F}_i$, and S_i. By setting

$$\bar{F}_i = \sum F^{i,\bar{m},m_1,m_2}_{i_1,i_2,j_1,j_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \cos(i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2), \tag{9}$$

we will have

/8

$$\dot{F}_i = \sum F^{i,\bar{m},m_1,m_2}_{\iota p, -\iota(p+q), j_1, j_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \cos(\iota p y + j_1 \omega_1 + j_2 \omega_2), \tag{10}$$

$$S_i = \sum{}' \frac{F^{i,\bar{m},m_1,m_2}_{i_1,i_2,j_1,j_2}}{i_1 n_1 + i_2 n_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \sin(i_1 y_1 + i_2 y_2 + j_1 \omega_1 + j_2 \omega_2). \tag{11}$$

The conditions satisfied by the indices are

$$|j_1| \underline{\le\le} m_1, \qquad |j_2| \underline{\le\le} m_2 = \text{even},$$
$$|i_1 + i_2 - j_1 - j_2| \underline{\le\le} \bar{m} + 2i - 2$$

for $\bar{F}_i$ and S_i, and

$$|j_1| \underline{\le\le} m_1, \qquad |j_2| \underline{\le\le} m_2 = \text{even},$$
$$|\iota q + j_1 + j_2| \underline{\le\le} \bar{m} + 2i - 2 \tag{12}$$

for $\dot{F}_1$.

In the sum Σ' which gives S_1, all terms where $i_1(p + q) + i_2 p = 0$ are excluded, i.e., the terms where $i_1 = \iota p$, $i_2 = -\iota(p + q)$, with ι being any integer.

Evidently, the variable x_2 does not enter the expansions (9), (10), and (11).

The coefficients

$$F^{i,\overline{m},m_1,m_2}_{i_1,i_2,j_1,j_2} \qquad (i = 2, 3, 4, \ldots),$$

defined in this Section must not be confused with the analogous coefficients introduced in Section 3. The two series of coefficients would be identical if, in the expansions in this Section, we would have given only a value of 0 to the index ι.

Let us now study the canonical transformation (2) in more detail. This transformation can be written in the form of

$$\begin{aligned} x_1 - \dot{x}_1 &= \frac{d(S - S_0)}{dy_1}, \qquad y_1 - \dot{y}_1 = -\frac{d(S - S_0)}{d\dot{x}_1}, \\ \xi_k - \dot{\xi}_k &= \frac{d(S - S_0)}{d\eta_k}, \qquad \eta_k - \dot{\eta}_k = -\frac{d(S - S_0)}{d\dot{\xi}_k}. \end{aligned} \tag{13}$$

On the one hand, we have excluded the relation which yields $x_2 - \dot{x}_2$, since we no longer need the auxiliary variable x_2 and, on the other hand, the relation $y_2 - \dot{y}_2 = 0$ which shows that /9

$$\dot{y}_2 = y_2 = t. \tag{14}$$

On solving eqs.(13) with respect to the variables x_1, y_1, ξ_k, η_k and on putting

$$\dot{\xi}_k = \dot{\varrho}_k \cos\dot{\omega}_k, \qquad \dot{\eta}_k = \dot{\varrho}_k \sin\dot{\omega}_k, \qquad (k = 1, 2), \tag{15}$$

we find that the differences $x_1 - \dot{x}_1$, $y_1 - \dot{y}_1$, $\xi_k - \dot{\xi}_k$, $\eta_k - \dot{\eta}_k$ can be expanded in the form

$$\sum C \mu^i e^{\overline{m}} \dot{\varrho}_1^{m_1} \dot{\varrho}_2^{m_2} \begin{matrix}\cos\\ \sin\end{matrix} (i_1 \dot{y}_1 + i_2 t + j_1 \dot{\omega}_1 + j_2 \dot{\omega}_2). \tag{16}$$

Here, we have cos for $x_1 - \dot{x}_1$ and $\xi_k - \dot{\xi}_k$ while we have sin for $y_1 - \dot{y}_1$ and $\eta_k - \dot{\eta}_k$. In addition, we have

$$|j_k| \leq m_k, \qquad (k = 1, 2).$$

Besides, m_2 and j_2 are even in the expansions of $x_1 - \dot{x}_1$, $y_1 - \dot{y}_1$, $\xi_1 - \dot{\xi}_1$,

$\eta_1 - \dot{\eta}_1$ but odd in the series that yield $\xi_2 - \dot{\xi}_2$ and $\eta_2 - \dot{\eta}_2$. Finally, we have

$$|i_1 + i_2 - j_1 - j_2| \leq\leq \bar{m} + 2i - 2$$

in the expansions yielding $x_1 - \dot{x}_1$ and $y_1 - \dot{y}_1$, but

$$|i_1 + i_2 - j_1 - j_2| \leq\leq \bar{m} + 2i - 1$$

in the expansions yielding $\xi_k - \dot{\xi}_k$ and $\eta_k - \dot{\eta}_k$.

Let us finally return to eq.(3). The characteristic function $\dot{F}$ can be thrown into the form

$$\dot{F} = \dot{F}_0 + \mu \dot{F}_1 + \mu^2 \dot{F}_2 + \cdots, \tag{17}$$

where

$$\dot{F}_0 = \frac{1}{2\dot{x}_1^2} - \dot{x}_2 \tag{18}$$

and /10

$$\dot{F}_i = \sum F^{i,\bar{m},m_1,m_2}_{\iota p, -\iota(p+q), j_1, j_2} e^{\bar{m}} \dot{\varrho}_1^{m_1} \dot{\varrho}_2^{m_2} \cos(\iota p\dot{y} + j_1 \dot{\omega}_1 + j_2 \dot{\omega}_2). \tag{19}$$

Here, we have put

$$p\dot{y} = p\dot{y}_1 - (p+q)\dot{y}_2 = p\dot{y}_1 - (p+q)t. \tag{20}$$

The relations (12) still remain valid.

Since $\dot{F}$ depends on $\dot{y}_1$ and $\dot{y}_2$ only in the combination $p\dot{y}$, we have the first integral

$$\dot{x}_2 + \frac{p+q}{p}\dot{x}_1 = \dot{C}.$$

It is thus easy to reduce the system (3) to three degrees of freedom. For this, it is sufficient to replace the variables

$$\begin{matrix} \dot{x}_1, & \dot{x}_2, \\ \dot{y}_1, & \dot{y}_2, \end{matrix}$$

by the new variables

$$\begin{matrix} \dot{x}_1, & C = \dot{x}_2 + \dfrac{p+q}{p}\dot{x}_1, \\ \dot{y} = \dot{y}_1 - \dfrac{p+q}{p}\dot{y}_2, & \dot{y}_2. \end{matrix} \tag{21}$$

This transformation is canonical. The new system becomes

$$\frac{d\dot{x}_1}{dt}=\frac{d\dot{F}}{d\dot{y}},\quad \frac{d\dot{y}}{dt}=-\frac{d\dot{F}}{d\dot{x}_1},$$
$$\frac{d\dot{\xi}_k}{dt}=\frac{d\dot{F}}{d\dot{\eta}_k},\quad \frac{d\dot{\eta}_k}{dt}=-\frac{d\dot{F}}{d\dot{\xi}_k},\quad (k=1,2). \tag{22}$$

In the expansion (17) of the characteristic function $\dot{F}$, we now have, because of eqs.(18) and (21),

$$\dot{F}_0=\frac{1}{2\dot{x}_1^2}+\frac{p+q}{p}\dot{x}_1-C. \tag{23}$$

The solution of the problem does not depend on the parameter C which finally appears only as an additive constant to the auxiliary variable x_2. /11

Section 17.

The method of reduction given in the preceding Section is of broad generality. This method is applicable no matter how small the quantity

$$n_1-\frac{p+q}{p} \tag{1}$$

might be. In addition, we have made no assumptions as to the magnitude of the eccentricity and of the inclination. Therefore, the discussed method is applicable not only to the case of minor planets but also to periodic comets whose mean motion is more or less at a commensurable and simple ratio to that of Jupiter. However, it is necessary to assume also here that the two orbits do not intersect.

To continue the reduction of the system (22) of Section 16, we will limit the problem by assuming, in this Part III of our research, that the quantity (1) is not too small but comparable in magnitude to $\mu^{1/2}$ and, on the other hand, that the eccentricities and the inclination are small, for example, of the order of $\mu^{1/2}$ or else of the order of $\mu^{1/4}$. Thus, the theory we will discuss here is that of minor planets designated as "characteristic" planets.

We can then put

$$-\frac{d\dot{F}_0}{dx_1}\equiv\frac{1}{\dot{x}_1^3}-\frac{p+q}{p}=\mu^{1/2}\Delta, \tag{2}$$

where the quantity Δ is comparable in magnitude to unity.

To reduce the canonical system (22) of Section 16 to two degrees of freedom, we will start from the equation

$$\dot F\left(\frac{dS}{d\dot y}, \dot y; \frac{dS}{d\dot\eta_k}, \dot\eta_k\right) = F^*\left(\dot x_1; \dot\xi_k, \frac{dS}{d\dot\xi_k}\right). \tag{3}$$

We will attempt to satisfy this equation by introducing for the unknown functions S and F^*, expansions of the form /12

$$F^* = F^*_0 + \cdot + \mu^{2/2} F^*_1 + \mu^{3/2} F^*_2 + \mu^{4/2} F^*_3 + \cdots,$$

$$S = S_0 + \mu^{1/2} S_1 + \mu^{2/2} S_2 + \mu^{3/2} S_3 + \mu^{4/2} S_4 + \cdots. \tag{4}$$

On expanding the two members of eq.(3) in powers of $\mu^{1/2}$ and on equating the coefficients of the same powers of $\mu^{1/2}$, it will become possible to successively determine the various terms of the expansions (4).

By setting

$$F^*_0 = \dot F_0 = \frac{1}{2\dot x_1^2} + \frac{p+q}{p}\dot x_1 - C,$$

$$S_0 = \dot x_1 \dot y + \sum_{k=1}^{2} \dot\xi_k \dot\eta_k, \tag{5}$$

eq.(3) is satisfied for $\mu = 0$.

After this, eq.(3) can be written as

$$\dot F\left(\dot x_1 + \frac{d(S-S_0)}{d\dot y}, \dot y; \dot\xi_k + \frac{d(S-S_0)}{d\dot\eta_k}, \dot\eta_k\right)$$
$$= F^*\left(\dot x_1; \dot\xi_k, \dot\eta_k + \frac{d(S-S_0)}{d\dot\xi_k}\right). \tag{3'}$$

The term in $\mu^{1/2}$ vanishes automatically.

On equating the coefficients of μ in the two members of eq.(3'), we obtain the equation

$$-\Delta \frac{dS_1}{d\dot y} + \frac{3}{2\dot x_1^4}\left(\frac{dS_1}{d\dot y}\right)^2 + \dot F_1 = F^*_1.$$

This is an equation of the second degree with respect to $\frac{dS_1}{d\dot y}$. As solution, we will select the root

$$\Delta \frac{dS_1}{d\dot y} = \frac{\Delta^2 \dot x_1^4}{3} - \frac{\Delta^2 \dot x_1^4}{3}\left\{1 - \frac{6}{\Delta^2 \dot x_1^4}(\dot F_1 - F^*_1)\right\}^{1/2}. \tag{6}$$

It is necessary to select the function F_1^* such that the mean value of the second term, considered as a periodic function of $\dot{y}$, shall vanish.

Moreover, in accordance with the formulas (12) of Section 16, we know that the exponents $\bar{m}$, m_1 and m_2 as well as the number ι, which appear in any term of the function F_1, satisfy the condition /13

$$|\iota q| \leq \leq \bar{m} + m_1 + m_2. \tag{7}$$

Thus, in $\dot{F}_1$, the terms periodic in $\dot{y}$ are at least of the degree q with respect to the quantities e', $\dot{\rho}_1$ and $\dot{\rho}_2$. We can conclude from this that the difference $\dot{F}_1 - F_1^*$ must be at least of the degree q with respect to these same quantities. Thus, it is permissible to expand the square root of the expression (6) and to write

$$\begin{aligned} \Delta \frac{dS_1}{d\dot{y}} = (\dot{F}_1 - F_1^*) + \frac{3}{2\Delta^2 x_1^4}(\dot{F}_1 - F_1^*)^2 \\ + \frac{9}{2\Delta^4 x_1^8}(\dot{F}_1 - F_1^*)^3 + \frac{135}{8\Delta^6 x_1^{12}}(\dot{F}_1 - F_1^*)^4 + \cdots. \end{aligned} \tag{8}$$

In the known function $\dot{F}_1$ as well as in the unknown function F_1^*, we will group together the terms which are of the same degree with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. No matter what the number q might be, we then can write

$$\dot{F}_1 = \dot{F}_{1,0} + \dot{F}_{1,1} + \dot{F}_{1,2} + \cdots, \tag{9}$$

$$F_1^* = F_{1,0}^* + F_{1,1}^* + F_{1,2}^* + \cdots, \tag{10}$$

where $\dot{F}_{1k}$ and F_{1k}^* are of the degree k with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$.

Let f be any function periodic with respect to $\dot{y}$. In Sections 17 and 18, we will denote by

$$[f]$$

the mean value of f, i.e., the term independent of $\dot{y}$ in the trigonometric expansion of f. For abbreviation, we will also write

$$\{f\} = f - [f].$$

After this, so as to cause the mean value of the second term of eq.(8) to vanish, it is necessary to determine one by one the various terms of the expansion (10), on the basis of the following formulas: /14

$$F_{1,0}^* = [\dot{F}_{1,0}],$$

$$F_{1,1}^* = [\dot{F}_{1,1}],$$

$$F^*_{1,2} = [\dot F_{1,2}] + \frac{3}{2\Delta^2 \dot x_1^4}[(\dot F_{1,1} - F^*_{1,1})^2],$$

$$F^*_{1,3} = [\dot F_{1,3}] + \frac{3}{\Delta^2 \dot x_1^4}[(\dot F_{1,1} - F^*_{1,1})(\dot F_{1,2} - F^*_{1,2})] + \frac{9}{2\Delta^4 \dot x_1^8}[(\dot F_{1,1} - F^*_{1,1})^3],$$

$$\begin{aligned} F^*_{1,4} = [\dot F_{1,4}] &+ \frac{3}{2\Delta^2 \dot x_1^4}[(\dot F_{1,2} - F^*_{1,2})^2 \\ &+ 2(\dot F_{1,1} - F^*_{1,1})(\dot F_{1,3} - F^*_{1,3})] \\ &+ \frac{27}{2\Delta^4 \dot x_1^8}[(\dot F_{1,1} - F^*_{1,1})^2(\dot F_{1,2} - F^*_{1,2})] \\ &+ \frac{135}{8\Delta^6 \dot x_1^{12}}[(\dot F_{1,1} - F^*_{1,1})^4], \end{aligned} \tag{11}$$

. . . .

Because of the relations (12) of Section 16 for i = 1, we will have in $\dot F_1$

$$|j_1| \le\le m_1, \qquad |j_2| \le\le m_2 = \text{even}, \\ |\iota q + j_1 + j_2| \le\le \overline{m}, \\ |\iota q| \le\le \overline{m} + m_1 + m_2. \tag{12}$$

Hence,

$$[\dot F_{1,2k'+1}] \equiv 0, \qquad (k' = 0, 1, 2 \ldots).$$

We even have

$$\dot F_{1,2k'+1} \equiv 0, \qquad \text{if q is even}$$

We will put /15

$$F^*_1 = \sum f^{1,\overline{m},m_1,m_2}_{j_1,j_2} e^{\overline{m}} \dot\varrho_1^{m_1} \dot\varrho_2^{m_2} \cos(j_1 \dot w_1 + j_2 \dot w_2). \tag{13}$$

In view of eqs.(11) and (12) it is easy to demonstrate that the integers m_k, j_k, and $\overline{m}$ which appear in the expansion (13) of F_1^*, still satisfy the relations

$$|j_1| \le\le m_1, \qquad |j_2| \le\le m_2 = \text{even}, \\ |j_1 + j_2| \le\le \overline{m}. \tag{14}$$

It follows from this that $\overline{m} + m_1 + m_2$ is even and that

$$F^*_{1,2k'+1} \equiv 0, \qquad (k' = 0, 1, 2, \ldots).$$

In view of all this, eqs.(11) are simplified and can be written as follows:

$$F^*_{1,0} = [\dot{F}_{1,0}],$$
$$F^*_{1,1} = 0,$$
$$F^*_{1,2} = [\dot{F}_{1,2}] + \frac{3}{2\Delta^2 x_1^4}[\{\dot{F}_{1,1}\}^2],$$
$$F^*_{1,3} = 0,$$
$$\begin{aligned} F^*_{1,4} = [\dot{F}_{1,4}] &+ \frac{3}{2\Delta^2 x_1^4}[\{\dot{F}_{1,2}\}^2 + 2\{\dot{F}_{1,1}\}\{\dot{F}_{1,3}\}] + \\ &+ \frac{27}{2\Delta^4 x_1^8}[\{\dot{F}_{1,1}\}^2\{\dot{F}_{1,2}\}] + \frac{135}{8\Delta^6 x_1^{12}}[\{\{\dot{F}_{1,1}\}^2\}^2], \end{aligned} \tag{15}$$

. . . .

After thus having successively determined the terms of the expansion (10), it is possible to find the function S_1 by means of a quadrature, using eq.(8). By arranging the terms in accordance with their degree, we can put

$$S_1 = S_{1,1} + S_{1,2} + S_{1,3} + \cdots.$$

In addition, the function S_1 will have the form /16

$$S_1 = \sum S^{1,\overline{m},m_1,m_2}_{\iota,j_1,j_2} e'^{\overline{m}} \varrho_1^{m_1} \varrho_2^{m_2} \sin(\iota p y + j_1 \omega_1 + j_2 \omega_2).$$

Evidently, the relations (12) are valid for this function S_1. It follows from this that

$$S_{1,2k'+1} \equiv 0, \qquad \text{if } q \text{ is even}.$$

Let us now compare the coefficients of $\mu^{3/2}$ in the two members of eq.(3'). This will yield the condition

$$-\Delta\frac{dS_2}{dy} + \frac{3}{x_1^4}\frac{dS_1}{dy}\frac{dS_2}{dy} + \Phi_2 - F_2^* = 0, \tag{16}$$

on putting, for abbreviation,

$$\Phi_2 = -\frac{2}{x_1^5}\left(\frac{dS_1}{dy}\right)^3 + \frac{d\dot{F}_1}{dx_1}\frac{dS_1}{dy} + \sum_{k=1}^{2}\left(\frac{d\dot{F}_1}{d\xi_k}\frac{dS_1}{d\eta_k} - \frac{dF_1^*}{d\eta_k}\frac{dS_1}{d\xi_k}\right). \tag{17}$$

Thus, taking eq.(6) into consideration, we obtain

$$\varDelta \frac{dS_2}{dy} = -\frac{\varPhi_2 - F_2^*}{\left(1 - \frac{6}{\varDelta^2 x_1^4}(\dot{F}_1 - F_1^*)\right)^{1/2}}. \tag{18}$$

On expanding the known function Φ_2 as well as the unknown function F_2^*, we can group together the terms that have the same degree with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. We thus put

$$\varPhi_2 = \varPhi_{2,0} + \varPhi_{2,1} + \varPhi_{2,2} + \cdots, \tag{19}$$

$$F_2^* = F_{2,0}^* + F_{2,1}^* + F_{2,2}^* + \cdots, \tag{20}$$

where Φ_{2k} and F_{2k}^* are of the degree k. After this, the condition that the mean value of the second term of the equality (18) shall vanish is expressed by the equations

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$$\begin{aligned}
&[\varPhi_{2,0}] - F_{2,0}^* = 0,\\
&[\varPhi_{2,1}] - F_{2,1}^* + \frac{3}{\varDelta^2 x_1^4}[(\dot{F}_{1,1} - F_{1,1}^*)(\varPhi_{2,0} - F_{2,0}^*)] = 0,\\
&[\varPhi_{2,2}] - F_{2,2}^* + \frac{3}{\varDelta^2 x_1^4}[(\dot{F}_{1,1} - F_{1,1}^*)(\varPhi_{2,1} - F_{2,1}^*)\\
&\qquad\qquad + (\dot{F}_{1,2} - F_{1,2}^*)(\varPhi_{2,0} - F_{2,0}^*)]\\
&\qquad + \frac{27}{2\varDelta^4 x_1^8}[(\dot{F}_{1,1} - F_{1,1}^*)^2(\varPhi_{2,0} - F_{2,0}^*)] = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{21}$$

The function Φ_2 has the form

$$\varPhi_2 = \sum \varPhi_{\iota,j_1,j_2}^{2,\bar{m},m_1,m_2} e'^{\bar{m}} \dot{\varrho}_1^{m_1} \dot{\varrho}_2^{m_2} \cos(\iota p\dot{y} + j_1\dot{\omega}_1 + j_2\dot{\omega}_2). \tag{22}$$

Obviously, we here have

$$|j_1| \leq m_1, \qquad |j_2| \leq m_2 - \text{even},$$
$$|\iota q + j_1 + j_2| \leq \bar{m} + 2. \tag{23}$$

The function F_2^*, whose various terms are obtained by means of eqs.(21), has the form

$$F_2^* = \sum f_{j_1\,j_2}^{2,\overline{m},\,m_1,m_2}\, e'^{\overline{m}} \dot{\varrho}_1^{m_1} \dot{\varrho}_2^{m_2} \cos(j_1\dot{\omega}_1 + j_2\dot{\omega}_2). \tag{24}$$

The integers m_k , j_k , and $\overline{m}$ obviously satisfy the conditions

$$|j_1| \leq\leq m_1, \qquad |j_2| \leq\leq m_2 = \text{even}, \qquad |j_1 + j_2| \leq\leq \overline{m} + 2. \tag{25}$$

The relations (23) resp. (25) show that

$$[\Phi_{2,2k'+1}] \equiv 0$$

and that /18

$$F^*_{2,2k'+1} \equiv 0.$$

Thus, the formulas (21) are simplified and become

$$\begin{aligned} F^*_{2,0} &= [\Phi_{2,0}], \\ F^*_{2,1} &= 0, \\ F^*_{2,2} &= [\Phi_{2,2}] + \frac{3}{\Delta^2 x_1^4}\left[\{\dot{F}_{1,1}\}\{\Phi_{2,1}\} + \{\dot{F}_{1,2}\}\{\Phi_{2,0}\}\right] \\ &\quad + \frac{27}{2\Delta^4 x_1^8}\left[\{\dot{F}^2_{1,1}\}\{\Phi_{2,0}\}\right], \\ F^*_{2,3} &= 0, \\ &\ldots\ldots\ldots \end{aligned} \tag{26}$$

Thus knowing the function F_2^* it is possible to obtain the function S_2 by means of eq.(18) after a quadrature. We can put

$$S_2 = S_{2,0} + S_{2,1} + S_{2,2} + \cdots,$$

where $S_{2,k}$ is of the degree k with respect to e' , $\dot{\rho}_1$, and $\dot{\rho}_2$. In addition, the function S_2 has the form

$$S_2 = \sum S_{i,j_1,j_2}^{2,\overline{m},\,m_1,m_2}\, e'^{\overline{m}} \dot{\varrho}_1^{m_1} \dot{\varrho}_2^{m_2} \sin(i p \dot{y} + j_1\dot{\omega}_1 + j_2\dot{\omega}_2),$$

and it is obvious that the relations (23) are valid for all terms of S_2 . Hence

$$S_{2,2k'+1} \equiv 0, \qquad \text{if q is even.}$$

A comparison of the coefficients of μ^2 in the two members of eq.(3') will yield an equation of the form

$$-\Delta\frac{dS_3}{d\dot y}+\frac{3}{\dot x_1^4}\frac{dS_1}{d\dot y}\frac{dS_3}{d\dot y}+\Phi_3-F_3^*=0, \tag{27}$$

by putting, for abbreviation,

/19

$$\begin{aligned}
\Phi_3 = {} & \frac{3}{2\dot x_1^4}\left(\frac{dS_2}{d\dot y}\right)^2-\frac{6}{\dot x_1^5}\left(\frac{dS_1}{d\dot y}\right)^2\frac{dS_2}{d\dot y}+\frac{5}{2\dot x_1^6}\left(\frac{dS_1}{d\dot y}\right)^4\\
&+\frac{d\dot F_1}{d\dot x_1}\frac{dS_2}{d\dot y}+\sum_{k=1}^{2}\left(\frac{d\dot F_1}{d\dot\xi_k}\frac{dS_2}{d\dot\eta_k}-\frac{dF_1^*}{d\dot\eta_k}\frac{dS_2}{d\dot\xi_k}\right)\\
&+\frac{1}{2}\frac{d^2\dot F_1}{d\dot x_1^2}\left(\frac{dS_1}{d\dot y}\right)^2+\sum_{k=1}^{2}\frac{d^2\dot F_1}{d\dot x_1\,d\dot\xi_k}\frac{dS_1}{d\dot y}\frac{dS_1}{d\dot\eta_k}\\
&+\frac{1}{2}\sum_{k=1}^{2}\left(\frac{d^2\dot F_1}{d\dot\xi_k^2}\left(\frac{dS_1}{d\dot\eta_k}\right)^2-\frac{d^2F_1^*}{d\dot\eta_k^2}\left(\frac{dS_1}{d\dot\xi_k}\right)^2\right)\\
&+\frac{d^2\dot F_1}{d\dot\xi_1\,d\dot\xi_2}\frac{dS_1}{d\dot\eta_1}\frac{dS_1}{d\dot\eta_2}-\frac{d^2F_1^*}{d\dot\eta_1\,d\dot\eta_2}\frac{dS_1}{d\dot\xi_1}\frac{dS_1}{d\dot\xi_2}+\dot F_2.
\end{aligned}$$

Equation (27) has the same form as eq.(16). By treating this equation in the same manner, it is possible to derive the functions F_3^* and S_3.

We can continue in this manner and thus successively determine the various terms of the expansions (4).

We will have

$$F_i^*=\sum f_{j_1,j_2}^{i,\bar m,m_1,m_2}\,e'^{\bar m}\,\dot\varrho_1^{m_1}\,\dot\varrho_2^{m_2}\cos(j_1\dot\omega_1+j_2\dot\omega_2) \tag{28}$$

with the conditions

$$\begin{gathered}
|j_1|\leq\leq m_1,\qquad |j_2|\leq\leq m_2=\text{even},\\
|j_1+j_2|\leq\leq\bar m+2i-2
\end{gathered} \tag{29}$$

and

$$S_i=\sum S_{\iota,j_1,j_2}^{i,\bar m,m_1,m_2}\,e'^{\bar m}\,\dot\varrho_1^{m_1}\,\dot\varrho_2^{m_2}\cos(\iota p\dot y+j_1\dot\omega_1+j_2\dot\omega_2) \tag{30}$$

with the conditions

$$\begin{gathered}
|j_1|\leq\leq m_1,\qquad |j_2|\leq\leq m_2=\text{even},\\
|\iota q+j_1+j_2|\leq\leq\bar m+2i-2.
\end{gathered} \tag{31}$$

The function F_i^* is always even with respect to e', $\dot\rho_1$, and $\dot\rho_2$. The function S is even with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$ provided that q is an even number.

Now, we will introduce new variables x_1^*, y^*, ξ_k^*, η_k^*. For this purpose, we form the function $S(x_1^*, \dot{y}; \xi_k^*, \dot{\eta}_k)$ as well as the corresponding canonical transformation, which latter can be written as /20

$$\dot{x}_1 - x_1^* = \frac{d(S-S_0)}{d\dot{y}}, \qquad \dot{y} - y^* = -\frac{d(S-S_0)}{dx_1^*},$$
$$\dot{\xi}_k - \xi_k^* = \frac{d(S-S_0)}{d\dot{\eta}_k}, \qquad \dot{\eta}_k - \eta_k^* = -\frac{d(S-S_0)}{d\xi_k^*}, \quad (k=1,2). \tag{32}$$

By virtue of this transformation, we obtain

$$\dot{F}(\dot{x}_1, \dot{y}; \dot{\xi}_k, \dot{\eta}_k) = F^*(x_1^*; \xi_k^*, \eta_k^*).$$

This is directly obvious by writing x_1^*, ξ_k^* instead of $\dot{x}_1$, $\dot{\xi}_k$ in eq.(3').

The new variables satisfy the equations

$$\frac{d\xi_k^*}{dt} = \frac{dF^*}{d\eta_k^*}, \quad \frac{d\eta_k^*}{dt} = -\frac{dF^*}{d\xi_k^*}, \qquad (k=1,2) \tag{33}$$

$$\frac{dx_1^*}{dt} = 0, \quad \frac{dy^*}{dt} = -\frac{dF^*}{dx_1^*}. \tag{34}$$

This will yield the first integral

$$x_1^* = \text{const.}$$

We will then slightly modify the second equation of the system (34).

In Section 16, we first defined two arguments y and y_1, correlated by the relation

$$y_1 = y + \frac{p+q}{p} t, \tag{35}$$

and then two other arguments $\dot{y}$ and $\dot{y}_1$ such that

$$\dot{y}_1 = \dot{y} + \frac{p+q}{p} t. \tag{36}$$

Finally, by means of the transformation (32), we introduced the argument y^*.

Now, we will define a new argument y_1^* by putting /21

$$y_1^* = y^* + \frac{p+q}{p} t. \qquad (37)$$

The second equation of the system (34) will be substituted by the equation

$$\frac{dy_1^*}{dt} = \frac{p+q}{p} - \frac{dF^*}{dx_1^*} = x_1^{*-3} - \frac{d(F^* - F_0^*)}{dx_1^*}. \qquad (38)$$

Equations (33) form a canonical system with two degrees of freedom. After its integration, the expression for the variable y_1^* is obtained by means of a quadrature over eq.(38).

Equations (33) and (34) or (38) are known as equations of secular variations of the characteristic minor planets.

Let us return again to eqs.(32). By solving these equations for the variables $\dot{x}_1$, $\dot{y}$, $\dot{\xi}_k$, $\dot{\eta}_k$ and by putting

$$\xi_k^* = \varrho_k^* \cos \omega_k^*, \quad \eta_k^* = \varrho_k^* \sin \omega_k^*, \qquad (k=1,2), \qquad (39)$$

we find that the differences $\dot{x}_1 - x_1^*$, $\dot{y} - y^* \equiv \dot{y}_1 - y_1^*$, $\dot{\xi}_k - \xi_k^*$, $\dot{\eta}_k - \eta_k^*$ can be expanded in the form of

$$\sum C'(\sqrt{\mu})^i e^{\bar{m}} \varrho_1^{*m_1} \varrho_2^{*m_2} \begin{matrix}\cos\\ \sin\end{matrix} (\iota[py_1^* - (p+q)t] + j_1\omega_1^* + j_2\omega_2^*). \qquad (40)$$

Here, we have cos in the expressions of $\dot{x}_1 - x_1^*$ and $\dot{\xi}_k - \xi_k^*$, while we have sin in the expressions of $\dot{y}_1 - y_1^*$ and $\dot{\eta}_k - \eta_k^*$. We also have

$$|j_k| \le\le m_k, \qquad (k=1,2).$$

In addition, m_2 and j_2 are even in the expansions of $\dot{x}_1 - x_1^*$, $\dot{y}_1 - y_1^*$, $\dot{\xi}_1 - \xi_1^*$, $\dot{\eta}_1 - \eta_1^*$ but odd in the series that yield $\dot{\xi}_2 - \xi_2$ and $\dot{\eta}_2 - \eta_2^*$. Finally, we have

$$|\iota q + j_1 + j_2| \le\le \bar{m} + 2i - 2, \quad |\iota q| \le\le \bar{m} + m_1 + m_2 + 2i - 2$$

in the expansions yielding $\dot{x}_1 - x_1^*$ and $\dot{y}_1 - y_1^*$, but

$$|\iota q + j_1 + j_2| \le\le \bar{m} + 2i - 1, \quad |\iota q| \le\le \bar{m} + m_1 + m_2 + 2i - 1$$ /22

in the expansions yielding $\dot{\xi}_k - \xi_k$ and $\dot{\eta}_k - \eta_k^*$.

All this readily results from the second expansion (4) which yields S, from the formulas (30) and (31) which determine the form of the functions S_1, and from the relations (36) and (37).

Section 18.

We will study the principal terms of the expansions of the functions F^* and S in more detail, by successively considering the cases where $q \geq 4$, $q = 3$, $q = 2$, or $q = 1$.

We always have

$$F_0^* = \dot{F}_0 = \frac{1}{2\dot{x}_1^2} + \frac{p+q}{p}\dot{x}_1 - C.$$

The expressions of the other functions $\dot{F}_i$ $(i = 1, 2, 3, \ldots)$ generally differ, depending on the value of the number q. However, these functions are still defined by the formulas (19) and the conditions (12) of Section 16.

Let us first assume that

$$q \geq 4.$$

We then have, considering separately the groups of terms which are of degrees 0, 1, 2, ... with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$,

$$\dot{F}_{1,0} = F^{1,0,0,0}_{0,0,0,0},$$

$$\dot{F}_{1,1} = 0,$$

$$\dot{F}_{1,2} = F^{1,0,2,0}_{0,0,0,0}\,\dot{\varrho}_1^2 + F^{1,0,0,2}_{0,0,0,0}\,\dot{\varrho}_2^2 + 2F^{1,1,1,0}_{0,0,1,0}\,e'\dot{\varrho}_1\cos\dot{\omega}_1 + F^{1,2,0,0}_{0,0,0,0}\,e'^2,$$

$$\dot{F}_{1,3} = 0,$$

$$\begin{aligned}
[\dot{F}_{1,4}] = {} & F^{1,0,4,0}_{0,0,0,0}\,\dot{\varrho}_1^4 + F^{1,0,2,2}_{0,0,0,0}\,\dot{\varrho}_1^2\dot{\varrho}_2^2 + F^{1,0,0,4}_{0,0,0,0}\,\dot{\varrho}_2^4 \\
& + 2F^{1,0,2,2}_{0,0,2,-2}\,\dot{\varrho}_1^2\dot{\varrho}_2^2\cos(2\dot{\omega}_1 - 2\dot{\omega}_2) + 2F^{1,1,3,0}_{0,0,1,0}\,e'\dot{\varrho}_1^3\cos\dot{\omega}_1 \\
& + 2F^{1,1,1,2}_{0,0,1,0}\,e'\dot{\varrho}_1\dot{\varrho}_2^2\cos\dot{\omega}_1 + 2F^{1,1,1,2}_{0,0,1,-2}\,e'\dot{\varrho}_1\dot{\varrho}_2^2\cos(\dot{\omega}_1 - 2\dot{\omega}_2) \\
& + F^{1,2,2,0}_{0,0,0,0}\,e'^2\dot{\varrho}_1^2 + 2F^{1,2,2,0}_{0,0,2,0}\,e'^2\dot{\varrho}_1^2\cos 2\dot{\omega}_1 + F^{1,2,0,2}_{0,0,0,0}\,e'^2\dot{\varrho}_2^2 \\
& + 2F^{1,2,0,2}_{0,0,0,2}\,e'^2\dot{\varrho}_2^2\cos 2\dot{\omega}_2 + 2F^{1,3,1,0}_{0,0,1,0}\,e'^3\dot{\varrho}_1\cos\dot{\omega}_1 + F^{1,4,0,0}_{0,0,0,0}\,e'^4, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$\dot{F}_{2,0} = F^{2,0,0,0}_{0,0,0,0},$$ /23

$$\dot{F}_{2,1} = 0,$$

$$\begin{aligned}
[\dot{F}_{2,2}] = {} & F^{2,0,2,0}_{0,0,0,0}\,\dot{\varrho}_1^2 + 2F^{2,0,2,0}_{0,0,2,0}\,\dot{\varrho}_1^2\cos 2\dot{\omega}_1 + F^{2,0,0,2}_{0,0,0,0}\,\dot{\varrho}_2^2 \\
& + 2F^{2,0,0,2}_{0,0,0,2}\,\dot{\varrho}_2^2\cos 2\dot{\omega}_2 + 2F^{2,1,1,0}_{0,0,1,0}\,e'\dot{\varrho}_1\cos\dot{\omega}_1 + F^{2,2,0,0}_{0,0,0,0}\,e'^2, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Equation (8) of Section 17 shows that $\dot{F}_i - F_i^*$ is of the order q with respect to e', $\dot{\rho}$, and $\dot{\rho}_2$. Hence,

$$F^*_{1,0} = \dot{F}_{1,0},$$
$$F^*_{1,2} = \dot{F}_{1,2}, \qquad (for \quad q \geq 4) \qquad (1)$$
$$F^*_{1,4} = [\dot{F}_{1,4}].$$

In accordance with the same formula (8) of Section 17, the function S_1 is of the order q with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. Thus, eq.(17) of Section 17 shows that the function Φ_2 is of the order q and that $[\Phi_2]$ is of the order q + + 2. Therefore, eq.(18) of Section 17 whose denominator differs from unity by quantities of only the order q, demonstrates that

$$F^*_{2,0} = 0,$$
$$F^*_{2,2} = 0, \qquad (for \quad q \geq 4) \qquad (2)$$
$$F^*_{2,4} = 0.$$

We can then pass to the function S_2. According to eq.(18) of Section 17, the quantity S_2 is of the order q with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$.

In continuing, it is easy to demonstrate that the function $\Phi_3 - \dot{F}_2$ is of the order q and that $[\Phi_3 - \dot{F}_2]$ is of the order q + 2. The functions F_3^* and S_3 are obtained by an equation derived from eq.(18) of Section 17 by writing there S_3, Φ_3, F_3^* instead of S_2, Φ_2, F_2^*. Hence,

$$F^*_{3,0} = \dot{F}_{2,0},$$
$$F^*_{3,2} = [\dot{F}_{2,2}], \qquad (for \quad q \geq 4) \qquad (3)$$
$$F^*_{3,4} = [\dot{F}_{2,4}].$$

Then, we find immediately that the function S_3, as the function $\{\dot{F}_2\}$, is /24 of the order q - 2 with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$.

Finally, it is easy to demonstrate that the function Φ_4, of which we did not give the expression, is of the order q - 2 and that $[\Phi_4]$ is of the order q. Hence,

$$F^*_{4,0} = 0,$$
$$F^*_{4,2} = 0. \qquad (for \quad q \geq 4) \qquad (4)$$

Let us now assume that

$$q = 3.$$

.. this case, we must start from the formulas

$$\dot{F}_{1,0} = F^{1,0,0,0}_{0,0,0,0},$$

$$\dot{F}_{1,1} = 0,$$

$$\dot{F}_{1,2} = F^{1,0,2,0}_{0,0,0,0}\dot{\varrho}_1^2 + F^{1,0,0,2}_{0,0,0,0}\dot{\varrho}_2^2 + 2F^{1,1,1,0}_{0,0,1,0}e'\dot{\varrho}_1\cos\dot{\omega}_1 + F^{1,2,0,0}_{0,0,0,0}e'^2,$$

$$\begin{aligned}\dot{F}_{1,3} = {} & 2F^{1,0,3,0}_{-p,p+3,3,0}\dot{\varrho}_1^3\cos(p\dot{y}-3\dot{\omega}_1) \\ & + 2F^{1,0,1,2}_{-p,p+3,1,2}\dot{\varrho}_1\dot{\varrho}_2^2\cos(p\dot{y}-\dot{\omega}_1-2\dot{\omega}_2) \\ & + 2F^{1,1,2,0}_{-p,p+3,2,0}e'\dot{\varrho}_1^2\cos(p\dot{y}-2\dot{\omega}_1) \\ & + 2F^{1,1,0,2}_{-p,p+3,0,2}e'\dot{\varrho}_2^2\cos(p\dot{y}-2\dot{\omega}_2) \\ & + 2F^{1,2,1,0}_{-p,p+3,1,0}e'^2\dot{\varrho}_1\cos(p\dot{y}-\dot{\omega}_1) \\ & + 2F^{1,3,0,0}_{-p,p+3,0,0}e'^3\cos p\dot{y},\end{aligned}$$

$\dot{F}_{1,4} =$ expression of $[\dot{F}_{1,4}]$ for $q = 4$,

. . . .

$$\dot{F}_{2,0} = F^{2,0,0,0}_{0,0,0,0},$$

$$\dot{F}_{2,1} = 2F^{2,0,1,0}_{-p,p+3,1,0}\dot{\varrho}_1\cos(p\dot{y}-\dot{\omega}_1) + 2F^{2,1,0,0}_{-p,p+3,0,0}e'\cos p\dot{y},$$

$\dot{F}_{2,2} =$ expression of $[\dot{F}_{2,2}]$ for $q = 4$,

. . . .

Thus, eqs.(15) of Section 17 will quite simply yield /25

$$\left.\begin{aligned} F^*_{1,0} &= \dot{F}_{1,0}, \\ F^*_{1,2} &= \dot{F}_{1,2}, \qquad (\text{for } q = 3) \\ F^*_{1,4} &= \dot{F}_{1,4}. \end{aligned}\right\} \tag{5}$$

The function S_1 is of the third order with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. The same is true for the function Φ_2 in accordance with eq.(17) of Section 17. Here, we even see that the mean value $[\Phi_2]$ is of the fourth order. Thus, eqs.(26) of Section 17 show that

$$\left.\begin{aligned} F^*_{2,0} &= 0, \\ F^*_{2,2} &= 0. \qquad (\text{for } q = 3) \end{aligned}\right. \tag{6}$$

According to eq.(18) of Section 17, S_2 is of the third order with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$.

It then is easy to see that $\Phi_3 - \dot{F}_2$ is of the third order and that the mean value $[\Phi_3 - \dot{F}_2]$ is of the fourth order. Formulas completely analogous to the formulas (26) of Section 17 will then show that

$$
\begin{aligned}
F^{*}_{3,0} &= \dot{F}_{2,0}, \\
F^{*}_{3,2} &= \dot{F}_{2,2}.
\end{aligned}
\qquad (\text{for} \quad q = 3) \tag{7}
$$

The equation which yields S_3 indicates that this function is of the first order in e', $\dot{\rho}_1$, and $\dot{\rho}_2$. Finally, it will be found without any difficulty that Φ_4 is of the first order, from which it follows that

$$
F^{*}_{4,0} = 0. \qquad (\text{for} \quad q = 3) \tag{8}
$$

We will now treat the cases in which

$$
q = 2.
$$

The various parts of the function $\dot{F}$ will then be

$$
\begin{aligned}
\dot{F}_{1,0} &= F^{1,0,0,0}_{0,0,0,0}, \\
\dot{F}_{1,1} &= 0, \\
\dot{F}_{1,2} &= F^{1,0,2,0}_{0,0,0,0}\,\dot{\varrho}_1^2 + 2F^{1,0,2,0}_{-p,p+2,2,0}\,\dot{\varrho}_1^2 \cos(p\dot{y} - 2\dot{\omega}_1) \\
&\quad + F^{1,0,0,2}_{0,0,0,0}\,\dot{\varrho}_2^2 + 2F^{1,0,0,2}_{-p,p+2,0,2}\,\dot{\varrho}_2^2 \cos(p\dot{y} - 2\dot{\omega}_2) \\
&\quad + 2F^{1,1,1,0}_{0,0,1,0}\,e'\dot{\varrho}_1 \cos\dot{\omega}_1 + 2F^{1,1,1,0}_{-p,p+2,1,0}\,e'\dot{\varrho}_1 \cos(p\dot{y} - \dot{\omega}_1) \\
&\quad + F^{1,2,0,0}_{0,0,0,0}\,e'^2 + 2F^{1,2,0,0}_{-p,p+2,0,0}\,e'^2 \cos p\dot{y}, \\
\dot{F}_{1,3} &= 0, \\
[\dot{F}_{1,4}] &= \text{ expression of } [\dot{F}_{1,4}] \text{ for } q = 4, \\
&\quad \dots\dots\dots\dots\dots\dots \\
\dot{F}_{2,0} &= F^{2,0,0,0}_{0,0,0,0} + 2F^{2,0,0,0}_{-p,p+2,0,0} \cos p\dot{y}, \\
\dot{F}_{2,1} &= 0, \\
[\dot{F}_{2,2}] &= \text{ expression of } [\dot{F}_{2,2}] \text{ for } q = 4, \\
&\quad \dots\dots\dots\dots\dots\dots
\end{aligned}
$$

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Equations (15) of Section 17 will now lead to the expressions

$$
\begin{aligned}
F^{*}_{1,0} &= [\dot{F}_{1,0}] = F^{1,0,0,0}_{0,0,0,0}, \\
F^{*}_{1,2} &= [\dot{F}_{1,2}] = F^{1,0,2,0}_{0,0,0,0}\,\dot{\varrho}_1^2 + F^{1,0,0,2}_{0,0,0,0}\,\dot{\varrho}_2^2 + \\
&\qquad + 2F^{1,1,1,0}_{0,0,1,0}\,e'\dot{\varrho}_1 \cos\dot{\omega}_1 + F^{1,2,0,0}_{0,0,0,0}\,e'^2.
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
F_{1,4}^{*} &= [\dot{F}_{1,4}] + \frac{3}{2\Delta^2 \dot{x}_1^4}[\{F_{1,2}\}^2] \\
&= \left(F_{0,0,0,0}^{1,0,4,0} + \frac{3}{\Delta^2 \dot{x}_1^4}(F_{-p,p+2,2,0}^{1,0,2,0})^2\right)\dot{\varrho}_1^4 \\
&+ F_{0,0,0,0}^{1,0,2,2}\,\dot{\varrho}_1^2\dot{\varrho}_2^2 + \left(F_{0,0,0,0}^{1,0,0,4} + \frac{3}{\Delta^2 \dot{x}_1^4}(F_{-p,p+2,0,2}^{1,0,0,2})^2\right)\dot{\varrho}_2^4 \\
&+ 2\left(F_{0,0,2,-2}^{1,0,2,2} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,2,0}^{1,0,2,0}F_{-p,p+2,0,2}^{1,0,0,2}\right)\dot{\varrho}_1^2\dot{\varrho}_2^2\cos(2\dot{\omega}_1 - 2\dot{\omega}_2) \\
&+ 2\left(F_{0,0,1,0}^{1,1,3,0} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,2,0}^{1,0,2,0}F_{-p,p+2,1,0}^{1,1,1,0}\right)e'\dot{\varrho}_1^3\cos\dot{\omega}_1 \\
&+ F_{0,0,1,0}^{1,1,1,2}e'\dot{\varrho}_1\dot{\varrho}_2^2\cos\dot{\omega}_1 \\
&+ 2\left(F_{0,0,1,-2}^{1,1,1,2} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,0,2}^{1,0,0,2}F_{-p,p+2,1,0}^{1,1,1,0}\right)e'\dot{\varrho}_1\dot{\varrho}_2^2\cos(\dot{\omega}_1 - 2\dot{\omega}_2) \\
&+ \left(F_{0,0,0,0}^{1,2,2,0} + \frac{3}{\Delta^2 \dot{x}_1^4}(F_{-p,p+2,1,0}^{1,1,1,0})^2\right)e'^2\dot{\varrho}_1^2 + F_{0,0,0,0}^{1,2,0,2}e'^2\dot{\varrho}_2^2 \\
&+ 2\left(F_{0,0,2,0}^{1,2,2,0} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,2,0}^{1,0,2,0}F_{-p,p+2,0,0}^{1,2,0,0}\right)e'^2\dot{\varrho}_1^2\cos 2\dot{\omega}_1 \\
&+ 2\left(F_{0,0,0,2}^{1,2,0,2} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,0,2}^{1,0,0,2}F_{-p,p+2,0,0}^{1,2,0,0}\right)e'^2\dot{\varrho}_2^2\cos 2\dot{\omega}_2 \\
&+ 2\left(F_{0,0,1,0}^{1,3,1,0} + \frac{3}{\Delta^2 \dot{x}_1^4}F_{-p,p+2,1,0}^{1,1,1,0}F_{-p,p+2,0,0}^{1,2,0,0}\right)e'^3\dot{\varrho}_1\cos\dot{\omega}_1 \\
&+ \left(F_{0,0,0,0}^{1,4,0,0} + \frac{3}{\Delta^2 \dot{x}_1^4}(F_{-p,p+2,0,0}^{1,2,0,0})^2\right)e'^4.
\end{aligned}
\tag{9}
$$

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According to eq.(8) of Section 17, we immediately obtain the principal part of S_1 (i.e., the terms of second order) in the form of

$$
\begin{aligned}
p\Delta S_{1,2} &= 2F_{-p,p+2,2,0}^{1,0,2,0}\dot{\varrho}_1^2\sin(p\dot{y} - 2\dot{\omega}_1) \\
&+ 2F_{-p,p+2,0,2}^{1,0,0,2}\dot{\varrho}_2^2\sin(p\dot{y} - 2\dot{\omega}_2) \\
&+ 2F_{-p,p+2,1,0}^{1,1,1,0}e'\dot{\varrho}_1\sin(p\dot{y} - \dot{\omega}_1) \\
&+ 2F_{-p,p+2,0,0}^{1,2,0,0}e'^2\sin p\dot{y}.
\end{aligned}
\tag{10}
$$

Equation (17) of Section 17 indicates that

$$
\Phi_{2,c} = 0,
$$

$$
[\Phi_{2,2}] = \sum_{k=1}^{2}\left[\frac{d\{\dot{F}_{1,2}\}}{d\xi_k}\frac{dS_{1,2}}{d\eta_k}\right] = -\frac{1}{p\Delta}\sum_{k=1}^{2}\left[\left(\frac{d\{\dot{F}_{1,2}\}}{d\xi_k}\right)^2\right].
$$

Thus, eqs.(26) of Section 17 will yield

$$F^*_{2,0} = 0,$$

$$F^*_{2,2} = [\Phi_{2,2}] = -\frac{1}{p\Delta}\{8(F^{1,0,2,0}_{-p,p+2,2,0})^2\dot{\varrho}_1^2 + 8(F^{1,0,0,2}_{-p,p+2,0,2})^2\dot{\varrho}_2^2 \tag{11}$$

$$+ 8F^{1,0,2,0}_{-p,p+2,2,0}F^{1,1,1,0}_{-p,p+2,1,0}e'\dot{\varrho}_1\cos\dot{\omega}_1 + 2(F^{1,1,1,0}_{-p,p+2,1,0})^2 e'^2\}.$$

We can now pass to the function S_2. Since $\Psi_{2,0} = 0$, eq.(18) of Section 17 shows that the function S_2 is of the second order with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. Equations (17) and (18) of Section 17 readily yield /28

$$p^2\Delta^2 S_{2,2} = 2F^{1,0,2,0}_{-p,p+2,2,0}\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4F^{1,0,2,0}_{0,0,0,0}\right)\dot{\varrho}_1^2\sin(p\dot{y} - 2\dot{\omega}_1)$$

$$+ 2F^{1,0,0,2}_{-p,p+2,0,2}\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4F^{1,0,0,2}_{0,0,0,0}\right)\dot{\varrho}_2^2\sin(p\dot{y} - 2\dot{\omega}_2)$$

+ terms that cancel with e'

+ terms with $2p\dot{y}$ in the argument.

According to the definition of the function Φ_3, it is easy to see that

$$\Phi_{3,0} = \dot{F}_{2,0},$$

$$[\Phi_{3,2}] = [\dot{F}_{2,2}] + \sum_{k=1}^{2}\left[\frac{d\{\dot{F}_{1,2}\}}{d\dot{\xi}_k}\frac{dS_{2,2}}{d\dot{\eta}_k}\right].$$

Finally, formulas analogous to the formulas (26) of Section 17 will yield

$$F^*_{3,0} = [\Phi_{3,0}] = [\dot{F}_{2,0}] = F^{2,0,0,0}_{0,0,0,0},$$

$$F^*_{3,2} = [\Phi_{3,2}] + \frac{3}{\Delta^2\dot{x}_1^2}[\{\dot{F}_{1,2}\}\{\Phi_{3,0}\}]$$

$$= F^{2,0,2,0}_{0,0,0,0}\dot{\varrho}_1^2 + 2F^{2,0,2,0}_{0,0,2,0}\dot{\varrho}_1^2\cos 2\dot{\omega}_1$$

$$+ F^{2,0,0,2}_{0,0,0,0}\dot{\varrho}_2^2 + 2F^{2,0,0,2}_{0,0,0,2}\dot{\varrho}_2^2\cos 2\dot{\omega}_2$$

$$- \frac{8}{p^2\Delta^2}(F^{1,0,2,0}_{-p,p+2,2,0})^2\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4F^{1,0,2,0}_{0,0,0,0}\right)\dot{\varrho}_1^2 \tag{12}$$

$$- \frac{8}{p^2\Delta^2}(F^{1,0,0,2}_{-p,p+2,0,2})^2\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4F^{1,0,0,2}_{0,0,0,0}\right)\dot{\varrho}_2^2$$

$$+ \frac{6}{\Delta^2\dot{x}_1^2}F^{1,0,2,0}_{-p,p+2,2,0}F^{2,0,0,0}_{-p,p+2,0,0}\dot{\varrho}_1^2\cos 2\dot{\omega}_1$$

$$+ \frac{6}{\Delta^2\dot{x}_1^2}F^{1,0,0,2}_{-p,p+2,0,2}F^{2,0,0,0}_{-p,p+2,0,0}\dot{\varrho}_2^2\cos 2\dot{\omega}_2$$

+ terms that cancel with e'

Last, we will consider the most difficult case in which /29

$$q = 1.$$

Here, we restrict ourselves to merely deriving the formulas required for calculating the inequalities of the second order of magnitude, while considering $\sqrt{\mu}$, the eccentricities, and the inclination as being of the first order of magnitude. We will start from the expressions

$$\dot{F}_{1,0} = F^{1,0,0,0}_{0,0,0,0},$$

$$\dot{F}_{1,1} = 2F^{1,0,1,0}_{-p,p+1,1,0}\dot{\varrho}_1 \cos(p\dot{y} - \dot{\omega}_1) + 2F^{1,1,0,0}_{-p,p+1,0,0} e' \cos p\dot{y},$$

$$\begin{aligned}\dot{F}_{1,2} = {} & F^{1,0,2,0}_{0,0,0,0}\dot{\varrho}_1^2 + 2F^{1,0,2,0}_{-2p,2p+2,2,0}\dot{\varrho}_1^2 \cos(2p\dot{y} - 2\dot{\omega}_1) \\ & + F^{1,0,0,2}_{0,0,0,0}\dot{\varrho}_2^2 + 2F^{1,0,0,2}_{-2p,2p+2,0,2}\dot{\varrho}_2^2 \cos(2p\dot{y} - 2\dot{\omega}_2) \\ & + 2F^{1,1,1,0}_{0,0,1,0} e'\dot{\varrho}_1 \cos\dot{\omega}_1 + 2F^{1,1,1,0}_{-2p,2p+2,1,0} e'\dot{\varrho}_1 \cos(2p\dot{y} - \dot{\omega}_1) \\ & + F^{1,2,0,0}_{0,0,0,0} e'^2 + 2F^{1,2,0,0}_{-2p,2p+2,0,0} e'^2 \cos 2p\dot{y},\end{aligned}$$

$$\begin{aligned}\dot{F}_{1,3} = {} & 2F^{1,0,3,0}_{-p,p+1,1,0}\dot{\varrho}_1^3 \cos(p\dot{y} - \dot{\omega}_1) \\ & + 2F^{1,0,1,2}_{-p,p+1,1,0}\dot{\varrho}_1\dot{\varrho}_2^2 \cos(p\dot{y} - \dot{\omega}_1) \\ & + 2F^{1,0,1,2}_{-p,p+1,-1,2}\dot{\varrho}_1\dot{\varrho}_2^2 \cos(p\dot{y} + \dot{\omega}_1 - 2\dot{\omega}_2) \\ & + 2F^{1,1,2,0}_{-p,p+1,0,0} e'\dot{\varrho}_1^2 \cos p\dot{y} \\ & + 2F^{1,1,2,0}_{-p,p+1,2,0} e'\dot{\varrho}_1^2 \cos(p\dot{y} - 2\dot{\omega}_1) \\ & + 2F^{1,1,0,2}_{-p,p+1,0,0} e'\dot{\varrho}_2^2 \cos p\dot{y} \\ & + 2F^{1,1,0,2}_{-p,p+1,0,2} e'\dot{\varrho}_2^2 \cos(p\dot{y} - 2\dot{\omega}_2) \\ & + 2F^{1,2,1,0}_{-p,p+1,1,0} e'^2\dot{\varrho}_1 \cos(p\dot{y} - \dot{\omega}_1) \\ & + 2F^{1,2,1,0}_{-p,p+1,-1,0} e'^2\dot{\varrho}_1 \cos(p\dot{y} + \dot{\omega}_1) \\ & + 2F^{1,3,0,0}_{-p,p+1,0,0} e'^3 \cos p\dot{y}\end{aligned}$$

+ terms that contain 2 $p\dot{y}$ in the argument

According to eqs.(15) of Section 17, we will thus have /30

$$F^*_{1,0} = F^{1,0,0,0}_{0,0,0,0},$$

$$\begin{aligned}F^*_{1,2} = {} & \left(F^{1,0,2,0}_{0,0,0,0} + \frac{3}{J^2\dot{x}_1^4}(F^{1,0,1,0}_{-p,p+1,1,0})^2\right)\dot{\varrho}_1^2 + F^{1,0,0,2}_{0,0,0,0}\dot{\varrho}_2^2 \\ & + 2\left(F^{1,1,1,0}_{0,0,1,0} + \frac{3}{J^2\dot{x}_1^4}F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,0,0}_{-p,p+1,0,0}\right)e'\dot{\varrho}_1 \cos\dot{\omega}_1 \qquad (13)\end{aligned}$$

$$+\left(F^{1,2,0,0}_{0,0,0,0}+\frac{3}{\Delta^2 \dot{x}_1^4}\left(F^{1,1,0,0}_{-p,p+1,0,0}\right)^2\right)e'^2.$$

Equation (8) of Section 17 indicates that S_1 is of the first order with respect to e', $\dot{\rho}_1$, and $\dot{\rho}_2$. Obviously, we then have

$$p\Delta S_{1,1}=2F^{1,0,1,0}_{-p,p+1,1,0}\,\dot{\varrho}_1\sin(p\dot{y}-\dot{\omega}_1)+2F^{1,1,0,0}_{-p,p+1,0,0}\,e'\sin p\dot{y},$$

$$\begin{aligned}p\Delta S_{1,2}=&\left(F^{1,0,2,0}_{-2p,2p+2,2,0}+\frac{3}{2\Delta^2\dot{x}_1^4}\left(F^{1,0,1,0}_{-p,p+1,1,0}\right)^2\right)\dot{\varrho}_1^2\sin(2p\dot{y}-2\dot{\omega}_1)\\&+F^{1,0,0,2}_{-2p,2p+2,0,2}\,\dot{\varrho}_2^2\sin(2p\dot{y}-2\dot{\omega}_2)\\&+\left(F^{1,1,1,0}_{-2p,2p+2,1,0}+\frac{3}{\Delta^2\dot{x}_1^4}F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,0,0}_{-p,p+1,0,0}\right)e'\dot{\varrho}_1\sin(2p\dot{y}-\dot{\omega}_1)\\&+\left(F^{1,2,0,0}_{-2p,2p+2,0,0}+\frac{3}{2\Delta^2\dot{x}_1^4}\left(F^{1,1,0,0}_{-p,p+1,0,0}\right)^2\right)e'^2\sin 2p\dot{y}.\end{aligned}$$

Before continuing, we will derive, from eq.(17) of Section 17, more detailed formulas, namely,

$$\Phi_{2,0}=\frac{d\dot{F}_{1,1}}{d\dot{\xi}_1}\frac{dS_{1,1}}{d\dot{\eta}_1},$$

$$\Phi_{2,1}=\frac{d\dot{F}_{1,0}}{d\dot{x}_1}\frac{dS_{1,1}}{d\dot{y}}+\frac{d\dot{F}_{1,1}}{d\dot{\xi}_1}\frac{dS_{1,2}}{d\dot{\eta}_1}+\frac{d\dot{F}_{1,2}}{d\dot{\xi}_1}\frac{dS_{1,1}}{d\dot{\eta}_1}-\frac{d\dot{F}_{1,2}}{d\dot{\eta}_1}\frac{dS_{1,1}}{d\dot{\xi}_1},$$

$$\begin{aligned}[\Phi_{2,2}]=&\left[\frac{d\dot{F}_{1,1}}{d\dot{x}_1}\frac{dS_{1,1}}{d\dot{y}}\right]+\left[\frac{d\dot{F}_{1,1}}{d\dot{\xi}_1}\frac{dS_{1,3}}{d\dot{\eta}_1}\right]\\&+\sum_{k=1}^{2}\left[\frac{d\{\dot{F}_{1,2}\}}{d\dot{\xi}_k}\frac{dS_{1,2}}{d\dot{\eta}_k}\right]+\left[\frac{d\{\dot{F}_{1,3}\}}{d\dot{\xi}_1}\frac{dS_{1,1}}{d\dot{\eta}_1}\right].\end{aligned}$$

These formulas, taken together with the formulas (26) of Section 17, have /31
given the following expressions:

$$F^*_{2,0}=-\frac{2}{p\Delta}\left(F^{1,0,1,0}_{-p,p+1,1,0}\right)^2,$$

$$\begin{aligned}F^*_{2,2}=\frac{2}{p\Delta}\Bigg\{&-2\left(F^{1,0,2,0}_{-2p,2p+2,2,0}\right)^2\\&+F^{1,0,1,0}_{-p,p+1,1,0}\left(p\frac{dF^{1,0,1,0}_{-p,p+1,1,0}}{d\dot{x}_1}-4F^{1,0,2,0}_{-p,p+1,1,0}\right)\\&+\frac{3}{\Delta^2\dot{x}_1^4}\left(F^{1,0,1,0}_{-p,p+1,1,0}\right)^2\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1}-2F^{1,0,2,0}_{0,0,0,0}\right.\end{aligned}\qquad(15)$$

$$-6F^{1,0,2,0}_{-2p,2p+2,2,0}\Big) - \frac{45}{2\Delta^4\dot{x}_1^8}(F^{1,0,1,0}_{-p,p+1,1,0})^4\Big\}\dot{\varrho}_1^2$$

$$-\frac{4}{p\Delta}\{(F^{1,0,0,2}_{-2p,2p+2,0,2})^2 + F^{1,0,1,0}_{-p,p+1,1,0}F^{1,0,1,2}_{-p,p+1,1,0}\}\dot{\varrho}_2^2$$

$$+\frac{2}{p\Delta}\Big\{p\frac{d}{d\dot{x}_1}(F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,0,0}_{-p,p+1,0,0})$$

$$-4F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,2,0}_{-p,p+1,2,0}$$

$$-2F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,2,0}_{-p,p+1,0,0} - 2F^{1,0,2,0}_{-2p,2p+2,2,0}F^{1,1,1,0}_{-2p,2p+2,1,0}$$

$$+\frac{3}{\Delta^2\dot{x}_1^4}F^{1,0,1,0}_{-p,p+1,1,0}\Big(2pF^{1,1,0,0}_{-p,p+1,0,0}\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1}$$

$$-2F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,1,0}_{0,0,1,0} - 2F^{1,0,2,0}_{0,0,0,0}F^{1,1,0,0}_{-p,p+1,0,0}$$

$$-3F^{1,0,1,0}_{-p,p+1,1,0}F^{1,1,1,0}_{-2p,2p+2,1,0} - 6F^{1,0,2,0}_{-2p,2p+2,2,0}F^{1,1,0,0}_{-p,p+1,0,0}\Big)$$

$$-\frac{45}{\Delta^4\dot{x}_1^8}(F^{1,0,1,0}_{-p,p+1,1,0})^3F^{1,1,0,0}_{-p,p+1,0,0}\Big\}e'\dot{\varrho}_1\cos\dot{\omega}_1$$

+ terms in e'^2.

To derive, finally, the expressions of $S_{2,0}$ and of $S_{2,1}$, we will start from the formulas

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$$\Delta\frac{dS_{2,0}}{d\dot{y}} = \{\Phi_{2,0}\},$$

$$\Delta\frac{dS_{2,1}}{d\dot{y}} = \{\Phi_{2,1}\} + \frac{3}{\Delta^2\dot{x}_1^4}\dot{F}_{1,1}\{\Phi_{2,0}\}$$

which result from eq.(18) of Section 17. In this manner, we find

$$S_{2,0} = -\frac{1}{p^2\Delta^2}(F^{1,0,1,0}_{-p,p+1,1,0})^2\sin 2p\dot{y},$$

$$S_{2,1} = \frac{1}{p^2\Delta^2}F^{1,0,1,0}_{-p,p+1,1,0}\Big\{2p\frac{dF^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4F^{1,0,2,0}_{0,0,0,0}$$

$$-6F^{1,0,2,0}_{-2p,2p+2,2,0} - \frac{9}{\Delta^2\dot{x}_1^4}(F^{1,0,1,0}_{-p,p+1,1,0})^2\Big\}\dot{\varrho}_1\sin(p\dot{y} - \dot{\omega}_1)$$

$$-\frac{1}{p^2\Delta^2}F^{1,0,1,0}_{-p,p+1,1,0}\Big\{2F^{1,0,2,0}_{-2p,2p+2,2,0} \qquad (16)$$

$$+\frac{3}{\Delta^2\dot{x}_1^4}(F^{1,0,1,0}_{-p,p+1,1,0})^2\Big\}\dot{\varrho}_1\sin(3p\dot{y} - \dot{\omega}_1)$$

+ terms multiplied by e'.

This will finally field, for the various values of q, the principal coefficients of the expansion of the function F^*, expressed by means of the coefficients of the expansion of the function F:

$$q \geq 4:$$

$$f_{j_1, j_2}^{1, \bar{m}, m_1, m_2} = F_{0, 0, j_1, j_2}^{1, \bar{m}, m_1, m_2}, \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 4;$$

$$f_{j_1, j_2}^{2, \bar{m}, m_1, m_2} = 0 \qquad , \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 4;$$

$$f_{j_1, j_2}^{3, \bar{m}, m_1, m_2} = F_{0, 0, j_1, j_2}^{2, \bar{m}, m_1, m_2}, \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 4;$$

$$f_{j_1, j_2}^{4, \bar{m}, m_1, m_2} = 0 \qquad , \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 2.$$

$$q = 3:$$ 133

$$f_{j_1, j_2}^{1, \bar{m}, m_1, m_2} = F_{0, 0, j_1, j_2}^{1, \bar{m}, m_1, m_2}, \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 4;$$

$$f_{j_1, j}^{2, \bar{m}, m_1, m_2} = 0 \qquad , \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 2;$$

$$f_{j_1, j_2}^{3, \bar{m}, m_1, m_2} = F_{0, 0, j_1, j_2}^{2, \bar{m}, m_1, m_2}, \quad \text{as long as} \quad \bar{m} + m_1 + m_2 \leq 2;$$

$$f_{0, 0}^{4, 0, 0, 0} = 0.$$

$$q = 2:$$

$$f_{j_1, j_2}^{1, \bar{m}, m_1, m_2} = F_{0, 0, j_1, j_2}^{1, \bar{m}, m_1, m_2}, \quad \text{aslong as} \quad \bar{m} + m_1 + m_2 \leq 2;$$

$$f_{0, 0}^{1, 0, 4, 0} = F_{0, 0, 0, 0}^{1, 0, 4, 0} + \frac{3}{\Delta^2 \dot{x}_1^4} (F_{-p, p+2, 2, 0}^{1, 0, 2, 0})^2,$$

$$f_{0, 0}^{1, 0, 2, 2} = F_{0, 0, 0, 0}^{1, 0, 2, 2},$$

$$f_{2, -2}^{1, 0, 2, 2} = F_{0, 0, 2, -2}^{1, 0, 2, 2} + \frac{3}{\Delta^2 \dot{x}_1^4} F_{-p, p+2, 2, 0}^{1, 0, 2, 0} F_{-p, p+2, 0, 2}^{1, 0, 0, 2},$$

$$f_{0, 0}^{1, 0, 0, 4} = F_{0, 0, 0, 0}^{1, 0, 0, 4} + \frac{3}{\Delta^2 \dot{x}_1^4} (F_{-p, p+2, 0, 2}^{1, 0, 0, 2})^2,$$

$$f_{1, 0}^{1, 1, 3, 0} = F_{0, 0, 1, 0}^{1, 1, 3, 0} + \frac{3}{\Delta^2 \dot{x}_1^4} F_{-p, p+2, 2, 0}^{1, 0, 2, 0} F_{-p, p+2, 1, 0}^{1, 1, 1, 0},$$

$$f_{1, 0}^{1, 1, 1, 2} = F_{0, 0, 1, 0}^{1, 1, 1, 2},$$

$$f_{1, -2}^{1, 1, 1, 2} = F_{0, 0, 1, -2}^{1, 1, 1, 2} + \frac{3}{\Delta^2 \dot{x}_1^4} F_{-p, p+2, 0, 2}^{1, 0, 0, 2} F_{-p, p+2, 1, 0}^{1, 1, 1, 0},$$

$$f_{0, 0}^{1, 2, 2, 0} = F_{0, 0, 0, 0}^{1, 2, 2, 0} + \frac{3}{\Delta^2 \dot{x}_1^4} (F_{-p, p+2, 1, 0}^{1, 1, 1, 0})^2.$$

$$f^{1,2,2,0}_{2,0} = F^{1,2,2,0}_{0,0,2,0} + \frac{3}{\Delta^2 \dot{x}_1^4} F^{1,0,2,0}_{-p,p+2,2,0} F^{1,2,0,0}_{-p,p+2,0,0},$$

$$f^{1,2,0,2}_{0,0} = F^{1,2,0,2}_{0,0,0,0},$$

$$f^{1,2,0,2}_{0,2} = F^{1,2,0,2}_{0,0,0,2} + \frac{3}{\Delta^2 \dot{x}_1^4} F^{1,0,0,2}_{-p,p+2,0,2} F^{1,2,0,0}_{-p,p+2,0,0},$$ 134

$$f^{1,3,1,0}_{1,0} = F^{1,3,1,0}_{0,0,1,0} + \frac{3}{\Delta^2 \dot{x}_1^4} F^{1,1,1,0}_{-p,p+2,1,0} F^{1,2,0,0}_{-p,p+2,0,0},$$

$$f^{1,4,0,0}_{0,0} = F^{1,4,0,0}_{0,0,0,0} + \frac{3}{\Delta^2 \dot{x}_1^4} (F^{1,2,0,0}_{-p,p+2,0,0})^2;$$

. . . .

$$f^{2,0,0,0}_{0,0} = 0;$$

$$f^{2,0,2,0}_{0,0} = -\frac{8}{p\Delta} (F^{1,0,2,0}_{-p,p+2,2,0})^2,$$

$$f^{2,0,0,2}_{0,0} = -\frac{8}{p\Delta} (F^{1,0,0,2}_{-p,p+2,0,2})^2,$$

$$f^{2,1,1,0}_{1,0} = -\frac{4}{p\Delta} F^{1,0,2,0}_{-p,p+2,2,0} F^{1,1,1,0}_{-p,p+2,1,0}.$$

$$f^{2,2,0,0}_{0,0} = -\frac{2}{p\Delta} (F^{1,1,1,0}_{-p,p+2,1,0})^2;$$

. . . .

$$f^{3,0,0,0}_{0,0} = F^{2,0,0,0}_{0,0,0,0};$$

$$f^{3,0,2,0}_{0,0} = F^{2,0,2,0}_{0,0,0,0} - \frac{8}{p^2\Delta^2} (F^{1,0,2,0}_{-p,p+2,2,0})^2 \left(p \frac{d F^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4 F^{1,0,2,0}_{0,0,0,0}\right),$$

$$f^{3,0,2,0}_{2,0} = F^{2,0,2,0}_{0,0,2,0} + \frac{3}{\Delta^2 \dot{x}_1^4} F^{1,0,2,0}_{-p,p+2,2,0} F^{2,0,0,0}_{-p,p+2,0,0},$$

$$f^{3,0,0,2}_{0,0} = F^{2,0,0,2}_{0,0,0,0} - \frac{8}{p^2\Delta^2} (F^{1,0,0,2}_{-p,p+2,0,2})^2 \left(p \frac{d F^{1,0,0,0}_{0,0,0,0}}{d\dot{x}_1} - 4 F^{1,0,0,2}_{0,0,0,0}\right),$$

$$f^{3,0,0,2}_{0,2} = F^{2,0,0,2}_{0,0,0,2} + \frac{3}{\Delta^2 \dot{x}_1^4} F^{1,0,0,2}_{-p,p+2,0,2} F^{2,0,0,0}_{-p,p+2,0,0},$$

. . . .

$q = 1:$

$$f^{1,0,0,0}_{0,0} = F^{1,0,0,0}_{0,0,0,0};$$

$$f^{1,0,2,0}_{0,0} = F^{1,0,2,0}_{0,0,0,0} + \frac{3}{\Delta^2 \dot{x}_1^4} (F^{1,0,1,0}_{-p,p+1,1,0})^2,$$

$$f^{1,0,0,2}_{0,0} = F^{1,0,0,2}_{0,0,0,0},$$

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$$f^{1,1,1,0}_{1,0} = F^{1,1,1,0}_{0,0,1,0} + \frac{3}{\Delta^2 x_1^4} F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,0,0}_{-p,p+1,0,0},$$

$$f^{1,2,0,0}_{0,0} = F^{1,2,0,0}_{0,0,0,0} + \frac{3}{\Delta^2 x_1^4} (F^{1,1,0,0}_{-p,p+1,0,0})^2;$$

. . . .

$$f^{2,0,0,0}_{0,0} = -\frac{2}{p\Delta} (F^{1,0,1,0}_{-p,p+1,1,0})^2;$$

$$\begin{aligned} f^{2,0,2,0}_{0,0} = \frac{2}{p\Delta}\Big\{ & -2(F^{1,0,2,0}_{-2p,2p+2,2,0})^2 \\ & + F^{1,0,1,0}_{-p,p+1,1,0}\left(p\frac{dF^{1,0,1,0}_{-p,p+1,1,0}}{dx_1} - 4F^{1,0,3,0}_{-p,p+1,1,0}\right) \\ & + \frac{3}{\Delta^2 x_1^4}(F^{1,0,1,0}_{-p,p+1,1,0})^2\left(p\frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1} - 2F^{1,0,2,0}_{0,0,0,0}\right. \\ & \left. - 6F^{1,0,2,0}_{-2p,2p+2,2,0}\right) - \frac{45}{2\Delta^4 x_1^8}(F^{1,0,1,0}_{-p,p+1,1,0})^4\Big\}, \end{aligned}$$

$$f^{2,0,0,2}_{0,0} = -\frac{4}{p\Delta}\{(F^{1,0,0,2}_{-2p,2p+2,0,2})^2 + F^{1,0,1,0}_{-p,p+1,1,0} F^{1,0,1,2}_{-p,p+1,1,0}\},$$

$$\begin{aligned} f^{2,1,1,0}_{1,0} = \frac{1}{p\Delta}\Big\{ & p\frac{d}{dx_1}(F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,0,0}_{-p,p+1,0,0}) \\ & - 4F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,2,0}_{-p,p+1,2,0} \\ & - 2F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,2,0}_{-p,p+1,0,0} - 2F^{1,0,2,0}_{-2p,2p+2,2,0} F^{1,1,1,0}_{-2p,2p+2,1,0} \\ & + \frac{3}{\Delta^2 x_1^4} F^{1,0,1,0}_{-p,p+1,1,0}\left(2pF^{1,1,0,0}_{-p,p+1,0,0}\frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1}\right. \\ & - 2F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,1,0}_{0,0,1,0} - 2F^{1,0,2,0}_{0,0,0,0} F^{1,1,0,0}_{-p,p+1,0,0} \\ & \left. - 3F^{1,0,1,0}_{-p,p+1,1,0} F^{1,1,1,0}_{-2p,2p+2,1,0} - 6F^{1,0,2,0}_{-2p,2p+2,2,0} F^{1,1,0,0}_{-p,p+1,0,0}\right) \\ & - \frac{45}{\Delta^4 x_1^8}(F^{1,0,1,0}_{-p,p+1,1,0})^3 F^{1,1,0,0}_{-p,p+1,0,0}\Big\}, \end{aligned}$$

. . . .

Section 19. /36

In Sections 19 - 22 we will assume, as in the theory of ordinary planets, that the eccentricity e' and the unknowns ξ_k^* and η_k^* are comparable in magnitude to $\mu^{1/2}$.

It is then convenient to group pairwise the terms of the expansion of F^*, by putting

$$F^* = F_0^* + \mu(F_1^* + \sqrt{\mu}\, F_2^*) + \mu^2(F_3^* + \sqrt{\mu}\, F_4^*) + \cdots$$
$$= F_0^* + \mu\, \mathfrak{F}_1^* + \mu^2 \mathfrak{F}_2^* + \cdots, \tag{1}$$

$$\mathfrak{F}_i^* = F_{2i-1}^* + \sqrt{\mu}\, F_{2i}^* = \sum \mathfrak{F}_{j_1,j_2}^{i,\bar{m},m_1,m_2} e'^{\bar{m}} \varrho_1^{m_1} \varrho_2^{m_2} \cos(j_1\omega_1 + j_2\omega_2), \tag{2}$$

$$\mathfrak{F}_{j_1,j_2}^{i,\bar{m},m_1,m_2} = f_{j_1,j_2}^{2i-1,\bar{m},m_1,m_2} + \sqrt{\mu}\, f_{j_1,j_2}^{2i,\bar{m},m_1,m_2}. \tag{3}$$

The expansions (1) and (2) are analogous to the expansions of F^* and F_i^* in the theory of ordinary planets (see Section 3). However, there are some differences:

In the case of ordinary planets, the indices i, $\bar{m}$, m_1, m_2, j_1, j_2 of the coefficients $F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2}$ satisfy the conditions (11) of Section 3. In the case of characteristic planets, the indices of the coefficients (3) definitely satisfy the conditions

$$|j_1| \leq m_1, \qquad |j_2| \leq m_2 = \text{even},$$
$$|j_1 + j_2| \leq \bar{m} + 4i - 2,$$

because of eqs.(29) of Section 17. It seems quite probable that, in complete generality,

$$|j_1 + j_2| \leq \bar{m} + 2i - 2$$

applies also to the coefficients (3). By extensive calculations, which it is useless to reproduce here, I actually found that this relation is exact, at least as long as

$$2i + \bar{m} + m_1 + m_2 \leq 8, \qquad \text{if } q \geq 2, \tag{4}$$

and at least as long as

$$2i + \bar{m} + m_1 + m_2 \leq 6, \qquad \text{if } q = 1. \tag{5}$$

In the case of ordinary planets, the quantity /37

$$F_{0,0,0,0}^{1,0,2,0} + F_{0,0,0,0}^{1,0,0,2}$$

is identically zero. For characteristic planets, the corresponding quantity

$$\mathfrak{F}_{0,0}^{1,0,2,0} + \mathfrak{F}_{0,0}^{1,0,0,2}$$

cancels out only if $q \geq 3$; this quantity is of the order of $\sqrt{\mu}$ if $q = 2$ and is comparable to unity if $q = 1$.

To obtain the function F^* which appears in eqs.(33) and (38) of Section 17, we must write

$$x_1^*,\ \varrho_k^*,\ \omega_k^*,\ \xi_k^*,\ \eta_k^*$$

instead of

$$x_1,\ \varrho_k,\ \omega_k,\ \xi_k,\ \eta_k$$

in the expressions for the various terms of F^* given until now. Consequently, it is necessary to calculate the various coefficients $F_{0,0,j_1,j_2}^{1,\bar{a},s_1,s_2}$, $f_{j_1,j_2}^{1,\bar{a},s_1,s_2}$, $\mathfrak{F}_{j_1,j_2}^{1,\bar{a},s_1,s_2}$ with the constant value x_1^* as well as the quantity Δ, using the formula

$$\frac{1}{x_1^{*2}} - \frac{p+q}{p} = \sqrt{\mu}\,\Delta. \tag{6}$$

Before integrating eqs.(33) and (38) of Section 17, we will subject them to several transformations.

Let us first consider the highly interesting particular solution in which ξ_k^* and η_k^* have constant values. Let

$$\xi_1^* = \bar{\xi}, \quad \eta_1^* = 0, \quad \xi_2^* = 0, \quad \eta_2^* = 0 \tag{7}$$

be this particular solution.

The quantity $\bar{\xi}$ satisfies the equation /38

$$\frac{dF^*}{d\xi_1^*} = 0, \tag{8}$$

in whose first term, the values (7) of the variables must be introduced. In the coefficients $\mathfrak{F}_{j_1,j_2}^{1,\bar{a},s_1,s_2}$ of the expansion (2), which are linear with respect to $\sqrt{\mu}$, we can consider $\sqrt{\mu}$ as a parameter independent of μ. Thus, the first member of eq.(8) is expanded in powers of μ. In addition, the coefficients of the various powers of μ are odd in $\bar{\xi}$ and e'. Thus, the ratio $\bar{\xi}:e'$ can be expanded in powers of e'^2 and μ.

By finally putting

$$e' = \sqrt{\mu}\, e_3, \qquad \bar{\xi} = \sqrt{\mu}\, \xi_0, \tag{9}$$

we can set

$$(\xi : e')^j = (\xi_0 : e_0)^j = p_0^{(j)} + p_1^{(j)}\mu + p_2^{(j)}\mu^2 + \cdots, \quad (j=1,2,3\ldots). \tag{10}$$

The coefficient $p_s^{(j)}$ in this expansion is a polynomial of the degree s with respect to e_0^2.

We obviously have

$$p_0^{(1)} = -\widetilde{\delta}_{1,0}^{1,1,1,0} : \widetilde{\delta}_{0,0}^{1,0,2,0}, \qquad p_0^{(j)} = (p_0^{(1)})^j. \tag{11}$$

In the theory of motion of characteristic minor planets, the particular solution (7) of eqs.(33) of Section 17 corresponds to a certain solution which depends only on two arbitrary constants and on two arguments linear with respect to time.

Now, we will introduce new variables by putting

$$\begin{aligned} \xi_1^* &= \bar{\xi} + \sqrt{\mu}\,\xi' = \sqrt{\mu}\,(\xi_0 + \xi'), & \eta_1^* &= \sqrt{\mu}\,\eta', \\ \xi_2^* &= \sqrt{\mu}\,\xi'', & \eta_2^* &= \sqrt{\mu}\,\eta''. \end{aligned} \tag{12}$$

In addition, we set /139

$$t_1 = \mu t, \tag{13}$$

$$\frac{1}{\mu^2}(F^* - C) = H, \tag{14}$$

where C is a constant selected in such a manner that H cancels out with the variables ξ', η', ξ'', η''.

The new canonical system, equivalent to eqs.(33) of Section 17, is written as

$$\begin{aligned} \frac{d\xi'}{dt_1} &= \frac{dH}{d\eta'}, & \frac{d\eta'}{dt_1} &= -\frac{dH}{d\xi'}, \\ \frac{d\xi''}{dt_1} &= \frac{dH}{d\eta''}, & \frac{d\eta''}{dt_1} &= -\frac{dH}{d\xi''}. \end{aligned} \tag{15}$$

Finally, we will put

$$\xi' = \varrho' \cos\omega', \qquad \eta' = \varrho' \sin\omega', \tag{16}$$

$$\xi''=\varrho''\cos\omega'', \qquad \eta''=\varrho''\sin\omega''.$$

In view of the expansion of the function F^*, it is obvious that the function H can be expanded in powers of μ. Thus, let

$$H = H^{(0)} + \mu H^{(1)} + \mu^2 H^{(2)} + \cdots. \tag{17}$$

The function $H^{(m)}$ has the form

$$H^{(m)} = \sum_{\substack{m',m''\\ j',j''}} H^{m,m',m''}_{j',j''}\,\varrho'^{m'}\varrho''^{m''}\cos(j'\omega' + j''\omega''). \tag{18}$$

This is an even polynomial of the degree 2m + 2 with respect to ξ', η', ξ'', η'', e_0. The coefficient $H^{m,m',m''}_{j',j''}$ is a polynomial in e_0.

As in Section 3, this leads to the general formula

$$H^{m,m',m''}_{j',j''} = \sum_{i,\overline{m},\alpha_1,\beta_1,s} \binom{\alpha_1}{\alpha'}\binom{\beta_1}{\beta'}\,\mathfrak{H}^{i,\overline{m},m_1,m''}_{j_1,j''}\,e_0^{\overline{m}+m_1-m'}\,p_s^{(m_1-m')}, \tag{19}$$

where, for abbreviation, we have put /40

$$m_1=\alpha_1+\beta_1, \qquad m'=\alpha'+\beta', \qquad m''=\alpha''+\beta'',$$

$$j_1=\alpha_1-\beta_1, \qquad j'=\alpha'-\beta', \qquad j''=\alpha''-\beta''.$$

The whole numbers i, $\overline{m}$, α_1, β_1, s in the sum (19) must assume values that satisfy the conditions

$$i\geq 1, \qquad \overline{m}\geq 0, \qquad \alpha_1\geq\alpha', \qquad \beta_1\geq\beta', \qquad s\geq 0,$$

$$\begin{aligned}|\alpha_1-\beta_1+\alpha''-\beta''| &\leq \overline{m}+2i-2=\\ &=2m+2-(\alpha_1+\beta_1+\alpha''+\beta'')-2s.\end{aligned} \tag{20}$$

In the sum (18), the whole numbers m', m'', j', j'' are subject to the conditions

$$\begin{gathered}2\leq m'+m''\leq 2m+2,\\ |j'|\leq\leq m', \qquad |j''|\leq\leq m''=\text{even}.\end{gathered} \tag{21}$$

Here, we also have

$$|j'+j''|+m'+m''\leq\leq 2m+2, \tag{22}$$

at least as long as

$$m = 0, 1, 2, \qquad \text{if } q \geq 2,$$

and at least for

$$m = 0, 1, \qquad \text{if } q = 1.$$

This results from the relations (20), considered together with the conditions (4) and (5).

Let us finally add that

$$H^{m,m',m''}_{-j',-j''} = H^{m,m',m''}_{j',j''} \tag{23}$$

because of the relation

$$\tilde{\mathfrak{F}}^{i,\bar{m},m_1,m_2}_{-j_1,-j_2} = \tilde{\mathfrak{F}}^{i,\bar{m},m_1,m_2}_{j_1,j_2}.$$

Below, we give the expressions for all coefficients of $H^{(0)}$ and of $H^{(1)}$, /41 derived from the general equation (19) and valid for all values of q:

$$\begin{aligned}
&H^{0,2,0}_{0,0} = \tilde{\mathfrak{F}}^{1,0,2,0}_{0,0}, \qquad H^{0,0,2}_{0,0} = \tilde{\mathfrak{F}}^{1,0,0,2}_{0,0};\\
&H^{1,4,0}_{0,0} = \tilde{\mathfrak{F}}^{1,0,4,0}_{0,0}, \qquad H^{1,0,4}_{0,0} = \tilde{\mathfrak{F}}^{1,0,0,4}_{0,0},\\
&H^{1,2,2}_{0,0} = \tilde{\mathfrak{F}}^{1,0,2,2}_{0,0}, \qquad H^{1,2,2}_{2,-2} = \tilde{\mathfrak{F}}^{1,0,2,2}_{2,-2},\\
&H^{1,3,0}_{1,0} = e_0 \left(\tilde{\mathfrak{F}}^{1,1,3,0}_{1,0} + 2\tilde{\mathfrak{F}}^{1,0,4,0}_{0,0} p^{(1)}_0\right),\\
&H^{1,1,2}_{1,-2} = e_0 \left(\tilde{\mathfrak{F}}^{1,1,1,2}_{1,-2} + 2\tilde{\mathfrak{F}}^{1,0,2,2}_{2,-2} p^{(1)}_0\right),\\
&H^{1,1,2}_{1,0} = e_0 \left(\tilde{\mathfrak{F}}^{1,1,1,2}_{1,0} + \tilde{\mathfrak{F}}^{1,0,2,2}_{0,0}\, p^{(1)}_0\right),\\
&H^{1,2,0}_{2,0} = \tilde{\mathfrak{F}}^{2,0,2,0}_{2,0} + e_0^2 \left(\tilde{\mathfrak{F}}^{1,2,2,0}_{2,0} + \tilde{\mathfrak{F}}^{1,1,3,0}_{1,0} p^{(1)}_0 + \tilde{\mathfrak{F}}^{1,0,4,0}_{0,0} p^{(2)}_0\right),\\
&H^{1,2,0}_{0,0} = \tilde{\mathfrak{F}}^{2,0,2,0}_{0,0} + e_0^2 \left(\tilde{\mathfrak{F}}^{1,2,2,0}_{0,0} + \tilde{\mathfrak{F}}^{1,1,3,0}_{1,0} p^{(1)}_0 + 4\tilde{\mathfrak{F}}^{1,0,4,0}_{0,0} p^{(2)}_0\right),\\
&H^{1,0,2}_{0,2} = \tilde{\mathfrak{F}}^{2,0,0,2}_{0,2} + e_0^2 \left(\tilde{\mathfrak{F}}^{1,2,0,2}_{0,2} + \tilde{\mathfrak{F}}^{1,1,1,2}_{-1,2} p^{(1)}_0 + \tilde{\mathfrak{F}}^{1,0,2,2}_{-2,2} p^{(2)}_0\right),\\
&H^{1,0,2}_{0,0} = \tilde{\mathfrak{F}}^{2,0,0,2}_{0,0} + e_0^2\, \tilde{\mathfrak{F}}^{1,2,0,2}_{0,0}.
\end{aligned} \tag{24}$$

We will then set

$$H^{(0)} = -\frac{\nu_0'}{2}\varrho'^2 - \frac{\nu_0''}{2}\varrho''^2 \tag{25}$$

with the notations

$$\nu'_0 = -2\,\tilde{\mathfrak{F}}_{0,0}^{1,0,2,0} = -2\,(f_{0,0}^{1,0,2,0} + \sqrt{\mu}\, f_{0,0}^{2,0,2,0}),$$

$$\nu''_0 = -2\,\tilde{\mathfrak{F}}_{0,0}^{1,0,0,2} = -2\,(f_{0,0}^{1,0,0,2} + \sqrt{\mu}\, f_{0,0}^{2,0,0,2}). \tag{26}$$

The expressions of the coefficients $f_{j_1,j_2}^{i,k,n_1,n_2}$, for various values of q, are given at the end of Section 18.

Moreover, it is well known that /42

$$F_{0,0,0,0}^{1,0,2,0} + F_{0,0,0,0}^{1,0,0,2} \equiv 0$$

(see Section 3 of Part I).

From this it follows that

$$\nu'_0 + \nu''_0 = 0, \qquad \text{if } q \geq 3; \tag{27}$$

$$\nu'_0 + \nu''_0 = \sqrt{\mu}\,\frac{16}{p\Delta}\,[(F_{-p,p+2,2,0}^{1,0,2,0})^2 + (F_{-p,p+2,0,2}^{1,0,0,2})^2], \quad \text{if } q = 2; \tag{28}$$

and that

$$\nu'_0 + \nu''_0 \text{ is comparable to unity if } q = 1. \tag{29}$$

The expression (25) of the function $H^{(0)}$ indicates that the canonical system (15) of the secular variations of characteristic minor planets enters in the general type considered in Section 1 at the beginning of this report.

It then remains to transform once more the equations of secular variations.

In eqs.(15), we will substitute ξ', η', ξ'', η'' by the variables φ', ψ', φ'', ψ'' by setting

$$\varphi' = \xi' + \sqrt{-1}\,\eta' = \varrho' e^{\sqrt{-1}\,\omega'},$$

$$\psi' = \xi' - \sqrt{-1}\,\eta' = \varrho' e^{-\sqrt{-1}\,\omega'}, \tag{30}$$

$$\varphi'' = \xi'' + \sqrt{-1}\,\eta'' = \varrho'' e^{\sqrt{-1}\,\omega''},$$

$$\psi'' = \xi'' - \sqrt{-1}\,\eta'' = \varrho'' e^{-\sqrt{-1}\,\omega''}.$$

The new unknowns satisfy the equations

$$\frac{d\varphi'}{dt_1} = -2\sqrt{-1}\,\frac{dH}{d\psi'}, \qquad \frac{d\psi'}{dt_1} = 2\sqrt{-1}\,\frac{dH}{d\varphi'},$$
$$\frac{d\varphi''}{dt_1} = -2\sqrt{-1}\,\frac{dH}{d\psi''}, \qquad \frac{d\psi''}{dt_1} = 2\sqrt{-1}\,\frac{dH}{d\varphi''}. \tag{31}$$

The function $H^{(m)}$ which enters as coefficient of μ^m in the expansion (17) is expressed by /43

$$H^{(m)} = \sum H^{m,m',m''}_{j',j''}\,\varphi'^{\alpha'}\,\psi'^{\beta'}\,\varphi''^{\alpha''}\,\psi''^{\beta''}. \tag{32}$$

For abbreviation, we have used there

$$m' = \alpha' + \beta', \qquad m'' = \alpha'' + \beta'',$$
$$j' = \alpha' - \beta', \qquad j'' = \alpha'' - \beta''. \tag{33}$$

In the sum (32), the nonnegative integers α', β', α'', β'' satisfy the conditions

$$2 \leq \alpha' + \beta' + \alpha'' + \beta'' \leq 2m + 2, \tag{34}$$

$$\alpha'' + \beta'' = \text{even}, \tag{35}$$

$$|\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' \leq 2m + 2, \tag{36}$$

where the last condition is valid at least for m = 0, 1, 2 if $q \geq 2$ and for at least m = 0, 1 if q = 1. These conditions are equivalent to the conditions (21) and (22).

We have, specifically,

$$H^{(0)} = H^{0,2,0}_{0,0}\,\varphi'\,\psi' + H^{0,0,2}_{0,0}\,\varphi''\,\psi''$$
$$= -\frac{\nu_0'}{2}\,\varphi'\,\psi' - \frac{\nu_0''}{2}\,\varphi''\,\psi''. \tag{37}$$

As in Section 4, we will introduce the arguments

$$w' = \nu' t_1 + \gamma', \qquad w'' = \nu'' t_1 + \gamma'', \tag{38}$$

where γ' and γ'' are two arbitrary constants while ν' and ν'' are two still unknown quantities. We also put

$$\varphi'_0 = \varrho'_0 e^{\sqrt{-1}\, w'}, \qquad \psi'_0 = \varrho'_0 e^{-\sqrt{-1}\, w'},$$
$$\varphi''_0 = \varrho''_0 e^{\sqrt{-1}\, w''}, \qquad \psi''_0 = \varrho''_0 e^{-\sqrt{-1}\, w''}, \tag{39}$$

where ρ'_0 and ρ''_0 are two arbitrary constants.

It is convenient to consider φ', ψ', φ'', ψ'' as a function of the independent variables φ'_0, ψ'_0, φ''_0, ψ''_0 instead of t_1. This will then yield /44

$$\frac{1}{\sqrt{-1}} \frac{d}{dt_1} = \nu' D' + \nu'' D''$$

with the symbolic notations

$$D' = \varphi'_0 \frac{d}{d\varphi'_0} - \psi'_0 \frac{d}{d\psi'_0}, \qquad D'' = \varphi''_0 \frac{d}{d\varphi''_0} - \psi''_0 \frac{d}{d\psi''_0}. \tag{40}$$

In view of this, eqs.(31) can be written as follows

$$-(\nu' D' + \nu'' D'')(\psi'_0 \varphi') - (\nu' - \nu'_0)\psi'_0 \varphi' = 2\psi'_0 \frac{d(H - H^{(0)})}{d\psi'},$$
$$(\nu' D' + \nu'' D'')(\varphi'_0 \psi') - (\nu' - \nu'_0)\varphi'_0 \psi' = 2\varphi'_0 \frac{d(H - H^{(0)})}{d\varphi'},$$
$$-(\nu'' D'' + \nu' D')(\psi''_0 \varphi'') - (\nu'' - \nu''_0)\psi''_0 \varphi'' = 2\psi''_0 \frac{d(H - H^{(0)})}{d\psi''},$$
$$(\nu'' D'' + \nu' D')(\varphi''_0 \psi'') - (\nu'' - \nu''_0)\varphi''_0 \psi'' = 2\varphi''_0 \frac{d(H - H^{(0)})}{d\varphi''}. \tag{41}$$

We will introduce the variables φ'', ψ'', φ'', ψ'' also in eq.(38) of Section 17. For this purpose, we set

$$\frac{1}{\mu^2} \frac{d}{dx_1^*}\left(F^* - F_0^* - \mu\, \mathfrak{F}_{0,0}^{1,0,0,0}\right) = G = \sum_{m=0}^{\infty} \mu^m G^{(m)}, \tag{42}$$

$$G^{(m)} = \sum G_{j',j''}^{m,m',m''} \varphi'^{\alpha'} \psi'^{\beta'} \varphi''^{\alpha''} \psi''^{\beta''} \tag{43}$$

with the notations (33).

Now, because of formula (42), eq.(38) of Section 17 can be written as

$$\frac{dy_1^*}{dt} = x_1^{?-3} - \mu \frac{d\mathfrak{F}_{0,0}^{1,0,0,0}}{dx_1^*} - \mu^2 G. \qquad (44)$$

Let us make some statements on the functions $G^{(m)}$, defined by the formula (42). /45

Evidently, $G^{(m)}$ is an even polynomial of the degree 2m + 2 with respect to e_0, φ', ψ', φ'', ψ''.

The coefficient $G_{j',j''}^{m,m',m''}$, which is a polynomial in e_0, is given by the general expression

$$G_{j',j''}^{m,m',m''} = \sum_{i,\overline{m},\alpha_1,\beta_1,s} \binom{\alpha_1}{\alpha'} \binom{\beta_1}{\beta'} \frac{d\mathfrak{F}_{j_1,j'}^{i,\overline{m},m_1,m''}}{dx_1^*} e_0^{\overline{m}+m_1-m'} p_s^{(m_1-m')}, \qquad (45)$$

analogous to eq.(19). The indices i, $\overline{m}$, α_1, β_1, and s take all values that satisfy the conditions (20).

In addition, the nonnegative integers α', β', α'', β'' also satisfy the conditions (35) and (36) which latter holds at least for m = 0, 1, 2 if q ≥ 2 and for m = 0, 1 if q = 1.

It is obvious that the functions H and G do not change on permuting φ' and ψ' as well as φ'' and ψ''.

Evidently, the expression of the function $G^{(0)}$ has the form

$$G^{(0)} = G_{0,0}^{0,0,0} + G_{1,0}^{0,1,0}(\varphi' + \psi') + G_{0,0}^{0,2,0}\varphi'\psi' + G_{0,0}^{0,0,2}\varphi''\psi''.$$

In accordance with formula (45), on putting there m = 0, the formulas given below are readily obtained (valid for all values of the integer q):

$$G_{0,0}^{0,0,0} = \frac{d\mathfrak{F}_{0,0}^{2,0,0,0}}{dx_1^*} + e_0^2\left(\frac{d\mathfrak{F}_{0,0}^{1,2,0,0}}{dx_1^*} + 2\frac{d\mathfrak{F}_{1,0}^{1,1,1,0}}{dx_1^*} p_0^{(1)} + \frac{d\mathfrak{F}_{0,0}^{1,0,2,0}}{dx_1^*} p_0^{(2)}\right),$$

$$G_{1,0}^{0,1,0} = e_0\left(\frac{d\mathfrak{F}_{1,0}^{1,1,1,0}}{dx_1^*} + \frac{d\mathfrak{F}_{0,0}^{1,0,2,0}}{dx_1^*} p_0^{(1)}\right),$$

$$G_{0,0}^{0,2,0} = \frac{d\mathfrak{F}_{0,0}^{1,0,2,0}}{dx_1^*},$$

$$G_{0,0}^{0,0,2} = \frac{d\mathfrak{F}_{0,0}^{1,0,0,2}}{dx_1^*}.$$

Since the general formula (3) as well as the expressions of the coefficients $f_{j_1,j_2}^{1,\bar{m},m_1,m_2}$, noted at the end of Section 18, are given, we will have /46

$$G_{0,0}^{0,2,0} + G_{0,0}^{0,0,2} \equiv 0, \qquad \text{if } q \geq 3.$$

Conversely, for $q = 2$, this sum is of the order of $\sqrt{\mu}$ while, for $q = 1$, the same quantity is comparable in magnitude to unity.

Section 20.

In the theory of secular variations of characteristic minor planets, we must differentiate three cases, depending on whether $q \geq 3$, $q = 2$, or $q = 1$.

First, we will consider the case where

$$q \geq 3.$$

We have seen that the quantities ν'_0 and ν''_0 then satisfy the fundamental relation

$$\nu'_0 + \nu''_0 = 0,$$

as in the case of ordinary minor planets.

Obviously, there is no basic difference between the theory of secular variations of ordinary minor planets and the corresponding theory of characteristic minor planets, for which $q \geq 3$. The two theories merely differ by the fact that, for characteristic planets, the coefficients $H_{j',j''}^{m,m',m''}$ and $G_{j',j''}^{m,m',m''}$ depend on $\sqrt{\mu}$ and that the relation (22) of Section 19 has been established only for $m = 0, 1, 2$. Thus, in the integration of the system (41) of Section 19, we can apply step by step the methods used in Section 4 for the case of ordinary minor planets.

The unknowns are expanded in whole powers of μ, as in eqs.(8) of Section 4. The expressions given by us for the coefficients of the various powers of μ in these expansions are still in force.

Let us recall specifically eqs.(12) and (12') of Section 4. Let us also recall that the functions $[\psi'_0\varphi'_1]$, $[\varphi'_0\psi'_1]$, $[\psi''_0\varphi''_1]$ and $[\varphi''_0\psi''_1]$ are zero.

The quantity $\nu'_1 + \nu''_1$, which is linear in e_0^2, ρ'^2_0, and ρ''^2_0, appears, raised to certain powers, in the denominators in the various terms of the mentioned expansions. /47

In addition, theorem 1 of Section 5 is obviously valid also for characteristic minor planets where $q \geq 3$. Conversely, it is impossible in this case to prove theorems 2, 3, and 4 of Section 5 since the relation (36) of Section 19 has not been generally proved for all values of m.

Making use of eqs.(12) of Section 19, we can return to the variables ξ_1^*, η_1^*, ξ_2^*, η_2^*. As in Section 6, it is convenient to introduce the moduli of eccentricity and inclination by means of the formulas

$$\varepsilon' = \sqrt{\mu}\,\varrho'_0, \qquad \varepsilon'' = \sqrt{\mu}\,\varrho''_0 \tag{1}$$

and to set also

$$\delta = \mu\,(\nu'_1 + \nu''_1). \tag{2}$$

We will consider ε', ε'', e', and $\sqrt{\mu}$ as being of the order of magnitude of one.

The quantity δ, which is of the second order of magnitude, is linear with respect to ε'^2, ε''^2, e'^2, and μ. In the case in which q = 3, the coefficients of $H^{(1)}$ that enter the expression of δ contain the quantity $\sqrt{\mu}$ implicitly and in a linear manner.

Now, the general solution of the system (33) of Section 17 (for q ≥ 3) assumes the form

$$\begin{aligned}
\xi_1^* &= \bar{\xi} + \varepsilon' \cos w' + \sum_{k=1}^{\infty} \sum_{j',j''} A_{j',j''}^{(2k+1)} \cos (j' w' + j'' w''),\\
\eta_1^* &= \qquad \varepsilon' \sin w' + \sum_{k=1}^{\infty} \sum_{j',j''} A_{j',j''}^{(2k+1)} \sin (j' w' + j'' w''),\\
\xi_2^* &= \qquad \varepsilon'' \cos w'' + \sum_{k=1}^{\infty} \sum_{j',j''} B_{j',j''}^{(2k+1)} \cos (j' w' + j'' w''),\\
\eta_2^* &= \qquad \varepsilon'' \sin w'' + \sum_{k=1}^{\infty} \sum_{j',j''} B_{j',j''}^{(2k+1)} \sin (j' w' + j'' w'').
\end{aligned} \tag{3}$$

The quantity $\bar{\xi}:e'$, defined by eq.(8) of Section 19, can be expanded in powers of e'^2 and μ. /48

The coefficients $A^{(2k+1)}$ and $B^{(2k+1)}$ (k = 1, 2, 3, ...) are of the order of magnitude of 2k + 1. They are rational and homogeneous functions with respect to the quantities ε', ε'', e', and $\sqrt{\mu}$. Only even powers of $\sqrt{\mu}$ are encountered here. Each denominator is a power δ^s of δ, where the superscript s is ≥ 0 and < 2k + 1 - 3 if $j' - j'' = \pm 1$, but will be ≤ 2k + 1 - 5 if $j' - j'' \neq \pm 1$. (Here, the plus sign refers to the coefficients A and the minus sign to the coefficients B.) The numerators of $A_{j',j''}^{(2k+1)}$ and $B_{j',j''}^{(2k+1)}$ are odd polynomials in ε', ε'', and e' which contain the factor $\varepsilon'^{|j'|}\varepsilon''^{|j''|}$ if $j' + j''$ is odd and the factor $\varepsilon'^{|j'|}\varepsilon''^{|j''|}$ if $j' + j''$ is even. The other factor of each of these numerators is a polynomial homogeneous in ε'^2, ε''^2, e'^2, and μ.

As in Section 6, we still have

$$A^{(2k+1)}_{+1,0} \equiv 0, \qquad B^{(2k+1)}_{0,+1} \equiv 0, \qquad (k = 1, 2, 3, \ldots).$$

In addition, the coefficients

$$A^{(2k+1)}_{0,0}, \qquad (k = 1, 2, 3, \ldots)$$

vanish if $\epsilon' = \epsilon'' = 0$, since the special solution obtained in that case must coincide with the particular solution (7) of Section 19.

Finally, the fractional inequalities that truly contain a divisor δ^8 are at least of the order of magnitude of five. The fractional coefficients of the order of magnitude 5 are

$$A^{(5)}_{3,2}, \quad A^{(5)}_{-1,-2}, \quad B^{(5)}_{2,3}, \quad B^{(5)}_{-2,-1}.$$

Let us now pass to eq.(44) of Section 19.

In the function G, we will replace φ', ψ', φ'', ψ'' by their expansions in powers of μ.

This will yield the expansion

$$G = \sum_{i=0}^{\infty} \mu^i g^{(i)} \tag{4}$$

where $g^{(i)}$ is rational in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0. As in Section 6 (pp.39-41 of Part I), we can demonstrate that the rational function $g^{(i)}$ is of the degree $2i + 2$ with respect to e'_0, φ'_0, ψ''_0, φ''_0, ψ''_0 and that, in its denominator, the quantity $\nu'_1 + \nu''_1$ enters raised to a power which is at most $2i - 2$ for terms where $j' - j'' = \pm 1$ and at most $2i - 4$ for the other terms. /49

In Section 6 (pp.41-44 of Part I), we gave, for the arguments of the various terms of the functions $g^{(i)}$, an upper limit for the sum $|j'| + |j''|$. In the actual case, it is no longer possible to prove the existence of this limit since the condition (36) of Section 19 has been proved only for m = 0, 1, 2.

Nevertheless, we can investigate the functions $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$ in more detail. We will use the denotation $\bar{G}^{(n)}$ for $G^{(n)}$ by writing φ'_0, ψ'_0, φ''_0, ψ''_0 instead of φ', ψ', φ'', ψ''. Thus, the wanted expressions of $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$ will become

$$g^{(0)} = \bar{G}^{(0)} = G^{0,0,0}_{0,0} + G^{0,1,0}_{1,0}(\varphi'_0 + \psi'_0) + G^{0,2,0}_{0,0}(\varphi'_0\psi'_0 - \varphi''_0\psi''_0),$$

$$g^{(1)} = \bar{G}^{(1)} + G^{0,1,0}_{1,0}\left(\frac{\psi'_0\varphi'_1}{\psi'_0} + \frac{\varphi'_0\psi'_1}{\varphi'_0}\right) \tag{5}$$

$$+ G^{0,2,0}_{0,0}(\psi'_0\varphi'_1 + \varphi'_0\psi'_1 - \psi''_0\varphi''_1 - \varphi''_0\psi''_1),$$

$$g^{(2)} = \bar{G}^{(2)} + \frac{d\,G^{(1)}}{\psi'_0\,d\varphi'_0}\psi'_0\varphi'_1 + \frac{d\,G^{(1)}}{\varphi'_0\,d\psi'_0}\varphi'_0\psi'_1$$
$$+ \frac{d\,G^{(1)}}{\psi''_0\,d\varphi''_0}\psi''_0\varphi''_1 + \frac{d\,G^{(1)}}{\varphi''_0\,d\psi''_0}\varphi''_0\psi''_1$$
$$+ G^{0,1,0}_{1,0}\left(\frac{\{\psi'_0\varphi'_2\}}{\psi'_0} + \frac{\{\varphi'_0\psi'_2\}}{\varphi'_0}\right)$$
$$+ G^{0,1,0}_{1,0}\left(\frac{[\psi'_0\varphi'_2]}{\psi'_0} + \frac{[\varphi'_0\psi'_2]}{\varphi'_0}\right) \tag{7}$$
$$+ G^{0,2,0}_{0,0}\left(\frac{\psi'_0\varphi'_1\,\varphi'_0\psi'_1}{\varphi'_0\psi'_0} - \frac{\psi''_0\varphi''_1\,\varphi''_0\psi''_1}{\varphi''_0\psi''_0}\right)$$
$$+ G^{0,2,0}_{0,0}(\{\psi'_0\varphi'_2\} + \{\varphi'_0\psi'_2\} - \{\psi''_0\varphi''_2\} - \{\varphi''_0\psi''_2\}).$$

In $g^{(2)}$, one could also expect the term /50

$$G^{0,2,0}_{0,0}([\psi'_0\varphi'_2] + [\varphi'_0\psi'_2] - [\psi''_0\varphi''_2] - [\varphi''_0\psi''_2]),$$

but this long-period term vanishes identically. In fact, in the function H which is constant because of the first integral $H = h$, the long-period principal term which is nothing else but

$$H^{0,2,0}_{0,0}([\psi'_0\varphi'_2] + [\varphi'_0\psi'_2] - [\psi''_0\varphi''_2] - [\varphi''_0\psi''_2]),$$

must vanish.

Evidently, $g^{(0)}$ is a polynomial in φ'_0, ψ'_0, φ''_0, ψ''_0, while in its monomials $\varphi_0'^{\alpha'}\psi_0'^{\beta'}\varphi_0''^{\alpha''}\psi_0''^{\beta''}$ we always have

$$|\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' \leq 2. \tag{8}$$

Let us recall now that the long-period parts of the functions $(\psi'_0\varphi'_1)$ $(\varphi'_0\psi'_1)$, $(\psi''_0\varphi''_1)$, $(\varphi''_0\psi''_1)$ are zero and that the short-period parts are given by formulas entirely similar to the formulas (12') of Section 4. These four functions, like $\overline{H}^{(1)}$ and $\overline{G}^{(1)}$, are polynomials in φ'_0, ψ'_0, φ''_0, ψ''_0 whose exponents satisfy the condition

$$|\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' \leq 4. \tag{9}$$

This same relation is also valid for the polynomials φ'_1 and ψ'_1 since, on dividing $\psi'_0\varphi'_1$ by ψ'_0 and on dividing $\varphi'_0\psi'_1$ by φ'_0, the degree $\alpha' + \beta' + \alpha'' + \beta''$ of any term will diminish by one whereas $|\alpha' - \beta' + \alpha'' - \beta''|$ will be decreased or increased by one. Thus, we find that $g^{(1)}$ is a polynomial in φ'_0, ψ'_0, φ''_0, ψ''_0 which satisfies the condition (9).

Let us now pass to $g^{(2)}$. It is easy to demonstrate that the part of this function which does not depend on $[\psi'_0 \varphi'_2]$ and $[\varphi'_0 \psi'_2]$ is a polynomial in φ'_0, ψ'_0, φ''_0, ψ''_0 which satisfies the condition

$$|\alpha' - \beta' + \alpha'' - \beta''| + \alpha' + \beta' + \alpha'' + \beta'' \leq\leq 6. \tag{10}$$

Conversely, the part of $g^{(2)}$ which contains the functions $[\psi'_0 \varphi'_2]$ and $[\varphi'_0 \psi'_2]$ is rational with the denominator $(\nu'_1 + \nu''_1)^2$. Let us add here that, in this /51 part of $g^{(2)}$, we have $j' - j'' = \pm 1$ and, consequently, $j' \neq j''$.

Finally, we can conclude that no terms with very long period appear in $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$. In fact, for these terms we would have $j' = j'' = 2s > 0$ and, because of eqs. (8), (9), and (10),

$$8s \leq\leq 6,$$

from which it follows that

$$s = 0.$$

Now, to obtain the expression of the argument y_1^*, we will proceed exactly as in Section 6 (pp. 43-45 of Part I). This will lead to the expansion

$$y_1^* = nt + c + \sum_{k=1}^{r} \sum_{j',j''} C^{(2k)}_{j',j''} \sin(j'w' + j''w''),$$

which must be approached from the formulas (3). The coefficient $C^{(2k)}_{j',j''}$ is of the order of magnitude 2k; this coefficient is rational and homogeneous with respect to the quantities ϵ', ϵ'', e', and $\sqrt{\mu}$. Only even powers of $\sqrt{\mu}$ are encountered here. The denominator is a power of δ^s of δ, where the exponent s is ≥ 0 and

$$\begin{aligned} &\leq 2k-3, && \text{if } j' - j'' = 0;\\ &\leq 2k-4, && \text{if } j' - j'' = 1;\\ &\leq 2k-6, && \text{if } j' - j'' > 1. \end{aligned}$$

The numerator $C^{(2k)}_{j',j''}$ is an even polynomial in ϵ', ϵ'', and e' which contains the factor $\epsilon'^{|j'|} \epsilon''^{|j''|}$ if $j' - j''$ is even and the factor $e' \epsilon'^{|j'|} \epsilon''^{|j''|}$ if $j' - j''$ is odd. The other factor of the numerator is a polynomial homogeneous in ϵ'^2, ϵ''^2, e'^2, and μ. According to what we know of $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$, the coefficients $C^{(2)}_{j',j''}$ and $C^{(4)}_{j',j''}$ are polynomials. The fractional inequalities of the longitude are at least of the order of magnitude of six. The fractional coefficients of the order of magnitude of six are $C^{(6)}_{3,2}$, $C^{(6)}_{-1,-2}$, and $C^{(6)}_{2,2}$. The first two coefficients originate in the rational part of the function $g^{(2)}$, while the third coefficient has its origin in $[g^{(3)}]$.

The quantity n is designated as the absolute mean motion of the planet /52 and can be expanded in the form of

$$n = n^{(0)} + n^{(2)} + n^{(4)} + \cdots. \tag{12}$$

We have, specifically,

$$n^{(0)} = x_1^{*-3}, \qquad n^{(2)} = -\mu \frac{dF^{1,0,0,0}_{0,0,0,0}}{dx_1^*},$$

$$n^{(4)} = -\mu^2[[g^{(0)}]] = -\mu^2(G^{0,0,0}_{0,0} + G^{0,2,0}_{0,0}(\varrho_*^2 - \varrho_0''^2))$$

$$= -\mu^2 \frac{dF^{2,0,0,0}_{0,0,0,0}}{dx_1^*} - \mu(\varepsilon'^2 - \varepsilon''^2)\frac{dF^{1,0,2,0}_{0,0,0,0}}{dx_1^*}$$

$$-\mu e'^2 \left\{\frac{dF^{1,2,0,0}_{0,0,0,0}}{dx_1^*} - 2\frac{dF^{1,1,1,0}_{0,0,1,0}}{dx_1^*}\frac{F^{1,1,1,0}_{0,0,1,0}}{F^{1,0,2,0}_{0,0,0,0}} + \frac{dF^{1,0,2,0}_{0,0,0,0}}{dx_1^*}\left(\frac{F^{1,1,1,0}_{0,0,1,0}}{F^{1,0,2,0}_{0,0,0,0}}\right)^2\right\},$$

exactly as for the ordinary minor planets. The quantity $n^{(2k)}$ $(k = 2, 3, \ldots)$ which has the expression $n^{(2k)} = -\mu^k[[g^{(k-2)}]]$ is of the order of magnitude $2k$. The quantity is rational and homogeneous with respect to ε'^2, ε''^2, e'^2 and μ. The denominator of $n^{(2k)}$ is a power δ^s where the exponent s satisfies the conditions $0 \le s \le 2k - 8$. Thus, $n^{(4)}$, $n^{(6)}$, $n^{(8)}$ are polynomials. In addition, the $n^{(2k)}$ $(k = 1, 2, 3, \ldots)$ obviously contain μ as a factor.

As in the case of ordinary minor planets, we must differentiate, for the characteristic minor planets with $q \ge 3$, between two categories: regular planets for which the divisor $\nu_1' + \nu_1''$ is comparable to unity and singular planets for which this quantity $\nu_1' + \nu_1''$ is small (of the order of $\sqrt{\mu}$ or less).

The integration method used in this Section is applicable only to ordinary planets.

Section 21.

In this Section, we will treat the case in which

$$q = 2.$$

We have seen that we then have /53

$$\frac{\nu'_0 + \nu''_0}{\sqrt{\mu}} = \frac{16}{p\Delta}[(F^{1,0,2,0}_{-p,p+2,2,0})^2 + (F^{1,0,0,2}_{-p,p+2,0,2})^2]. \tag{1}$$

Let us start from eqs.(41) of Section 19. Since the quantity $\nu_0' + \nu_0''$ is of the order of $\sqrt{\mu}$, the unknowns must be expanded in powers of $\sqrt{\mu}$. We set

$$
\begin{aligned}
\varphi' &= \varphi'_0 + (\mu^{1/2}\varphi'_{1/2}) + \mu\varphi'_1 + (\mu^{3/2}\varphi'_{3/2}) + \mu^2\varphi'_2 + \mu^{5/2}\varphi'_{5/2} + \cdots,\\
\psi' &= \psi'_0 + (\mu^{1/2}\psi'_{1/2}) + \mu\psi'_1 + (\mu^{3/2}\psi'_{3/2}) + \mu^2\psi'_2 + \mu^{5/2}\psi'_{5/2} + \cdots,\\
\varphi'' &= \varphi''_0 + (\mu^{1/2}\varphi''_{1/2}) + \mu\varphi''_1 + (\mu^{3/2}\varphi''_{3/2}) + \mu^2\varphi''_2 + \mu^{5/2}\varphi''_{5/2} + \cdots,\\
\psi'' &= \psi''_0 + (\mu^{1/2}\psi''_{1/2}) + \mu\psi''_1 + (\mu^{3/2}\psi''_{3/2}) + \mu^2\psi''_2 + \mu^{5/2}\psi''_{5/2} + \cdots,\\
\nu' &= \nu'_0 + (\mu^{1/2}\nu'_{1/2}) + \mu\nu'_1 + (\mu^{3/2}\nu'_{3/2}) + \mu^2\nu'_2 + (\mu^{5/2}\nu'_{5/2}) + \cdots,\\
\nu'' &= \nu''_0 + (\mu^{1/2}\nu''_{1/2}) + \mu\nu''_1 + (\mu^{3/2}\nu''_{3/2}) + \mu^2\nu''_2 + (\mu^{5/2}\nu''_{5/2}) + \cdots.
\end{aligned}
\tag{2}
$$

It will be shown below that the terms in parentheses vanish identically.

We will introduce the expansions (2) in eqs.(41) of Section 19 and expand the two terms in powers of $\sqrt{\mu}$. A comparison of the two components of the coefficients of the same power of $\sqrt{\mu}$ will yield a sequence of equations by means of which the various coefficients of the series (2) can be determined.

Let f be any function expandable in powers of φ'_0, ψ'_0, φ''_0, ψ''_0. In Section 4 (pp. 21 and 22 of Part I), we defined the parts [f], [[f]], and {f} of the function f.

Below, it will be useful to employ the symbolic notation

$$D = \nu'_0 D' + \nu''_0 D''. \tag{3}$$

One more remark should be made here: It is obvious that the derivative /54

$$D[f]$$

contains the factor $\sqrt{\mu}$ because of the relation (1).

In view of this, let us first consider the independent terms of μ in eqs.(41) of Section 19. These terms vanish because of the identities

$$D(\varphi'_0\psi'_0) \equiv 0, \qquad D(\varphi''_0\psi''_0) \equiv 0.$$

Let us then compare the terms multiplied by $\sqrt{\mu}$. We find the equations

$$
\begin{aligned}
-D\{\psi'_0\varphi'_{1/2}\} - \nu'_{1/2}\psi'_0\varphi'_0 &= 0,\\
D\{\varphi'_0\psi'_{1/2}\} - \nu'_{1/2}\varphi'_0\psi'_0 &= 0,\\
-D\{\psi''_0\varphi''_{1/2}\} - \nu''_{1/2}\psi''_0\varphi''_0 &= 0,\\
D\{\varphi''_0\psi''_{1/2}\} - \nu''_{1/2}\varphi''_0\psi''_0 &= 0.
\end{aligned}
$$

These are satisfied by putting

$$\nu'_{1/2} = \nu''_{1/2} = 0;$$

$$\{\psi'_0 \varphi'_{1/2}\} = \{\varphi'_0 \psi'_{1/2}\} = \{\psi''_0 \varphi''_{1/2}\} = \{\varphi''_0 \psi''_{1/2}\} = 0. \tag{4}$$

So that the terms with the factor μ shall vanish, the following equations must be satisfied:

$$\begin{aligned}
-D\{\psi'_0 \varphi'_1\} - \frac{1}{\sqrt{\mu}} D[\psi'_0 \varphi'_{1/2}] - \nu'_1 \psi'_0 \varphi'_0 &= 2\psi'_0 \frac{dH^{(1)}}{d\psi'_0}, \\
D\{\varphi'_0 \psi'_1\} + \frac{1}{\sqrt{\mu}} D[\varphi'_0 \psi'_{1/2}] - \nu'_1 \varphi'_0 \psi'_0 &= 2\varphi'_0 \frac{dH^{(1)}}{d\varphi'_0}, \\
-D\{\psi''_0 \varphi''_1\} - \frac{1}{\sqrt{\mu}} D[\psi''_0 \varphi''_{1/2}] - \nu''_1 \psi''_0 \varphi''_0 &= 2\psi''_0 \frac{dH^{(1)}}{d\psi''_0}, \\
D\{\varphi''_0 \psi''_1\} + \frac{1}{\sqrt{\mu}} D[\varphi''_0 \psi''_{1/2}] - \nu''_1 \varphi''_0 \psi''_0 &= 2\varphi''_0 \frac{dH^{(1)}}{d\varphi''_0}.
\end{aligned} \tag{5}$$

In view of the expression of $[H^{(1)}]$ which is a constant, eqs.(5) are satisfied by putting /55

$$\begin{aligned}
\nu'_1 &= -2\frac{d[[H^{(1)}]]}{\varphi'_0 d\psi'_0} = -4H_{0,0}^{1,4,0}\varphi'_0\psi'_0 - 2H_{0,0}^{1,2,2}\varphi''_0\psi''_0 - 2H_{0,0}^{1,2,0}, \\
\nu''_1 &= -2\frac{d[[H^{(1)}]]}{\varphi''_0 d\psi''_0} = -4H_{0,0}^{1,0,4}\varphi''_0\psi''_0 - 2H_{0,0}^{1,2,2}\varphi'_0\psi'_0 - 2H_{0,0}^{1,0,2};
\end{aligned} \tag{6}$$

$$[\psi'_0 \varphi'_{1/2}] = [\varphi'_0 \psi'_{1/2}] = [\psi''_0 \varphi''_{1/2}] = [\varphi''_0 \psi''_{1/2}] = 0; \tag{7}$$

$$\begin{aligned}
\{\psi'_0 \varphi'_1\} &= -\sum \frac{2\beta' H_{j',j''}^{1,m',m''}}{j'\nu'_0 + j''\nu''_0}\mathfrak{M}, \\
\{\varphi'_0 \psi'_1\} &= \sum \frac{2\alpha' H_{j',j''}^{1,m',m''}}{j'\nu'_0 + j''\nu''_0}\mathfrak{M}, \\
\{\psi''_0 \varphi''_1\} &= -\sum \frac{2\beta'' H_{j',j''}^{1,m',m''}}{j'\nu'_0 + j''\nu''_0}\mathfrak{M}, \\
\{\varphi''_0 \psi''_1\} &= \sum \frac{2\alpha'' H_{j',j''}^{1,m',m''}}{j'\nu'_0 + j''\nu''_0}\mathfrak{M}.
\end{aligned} \qquad (j' \neq j'') \tag{8}$$

As before and for abbreviation, we have set

$$\mathfrak{M} = \varphi_0'^{\alpha'} \psi_0'^{\beta'} \varphi_0''^{\alpha''} \psi_0''^{\beta''} = \varrho_0'^{m'} \varrho_0''^{m''} e^{\sqrt{-1}\,(j'w' + j''w'')}.$$

The functions (8) are even and of the degree 4 with respect to ε, φ_0', ψ_0', φ_0'', ψ_0''. The first two functions as well as the two last ones are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''.

The equality of the terms in $\mu^{3/2}$ in the two members of eqs.(41) of Section 19 is expressed by the equations

$$-D\{\psi'_0\, \varphi'_{3/2}\} - \frac{1}{\sqrt{\mu}}\, D[\psi'_0\, \varphi'_1] - \nu'_{3/2}\, \psi'_0\, \varphi'_0 = 0,$$

$$D\{\varphi'_0\, \psi'_{3/2}\} + \frac{1}{\sqrt{\mu}}\, D[\varphi'_0\, \psi'_1] - \nu'_{3/2}\, \varphi'_0\, \psi'_0 = 0,$$ /56

$$-D\{\psi''_0\, \varphi''_{3/2}\} - \frac{1}{\sqrt{\mu}}\, D[\psi''_0\, \varphi''_1] - \nu''_{3/2}\, \psi''_0\, \varphi''_0 = 0,$$

$$D\{\varphi''_0\, \psi''_{3/2}\} + \frac{1}{\sqrt{\mu}}\, D[\varphi''_0\, \psi''_1] - \nu''_{3/2}\, \varphi''_0\, \psi''_0 = 0.$$

To satisfy these equations, we must set

$$\begin{gathered} \nu'_{3/2} = \nu''_{3/2} = 0; \\ [\psi'_0\, \varphi'_1] = [\varphi'_0\, \psi'_1] = [\psi''_0\, \varphi''_1] = [\varphi''_0\, \psi''_1] = 0; \\ \{\psi'_0\, \varphi'_{3/2}\} = \{\varphi'_0\, \psi'_{3/2}\} = \{\psi''_0\, \varphi''_{3/2}\} = \{\varphi''_0\, \psi''_{3/2}\} = 0. \end{gathered} \tag{9}$$

Let us now proceed to the power μ^2. The corresponding equations are as follows:

$$\begin{aligned} -D\{\psi'_0\, \varphi'_2\} - \frac{1}{\sqrt{\mu}}\, D[\psi'_0\, \varphi'_{3/2}] - \nu'_2\, \psi'_0\, \varphi'_0 &= \psi'_0\, A'_2, \\ D\{\varphi'_0\, \psi'_2\} + \frac{1}{\sqrt{\mu}}\, D[\varphi'_0\, \psi'_{3/2}] - \nu'_2\, \varphi'_0\, \psi'_0 &= \varphi'_0\, B'_2, \\ -D\{\psi''_0\, \varphi''_2\} - \frac{1}{\sqrt{\mu}}\, D[\psi''_0\, \varphi''_{3/2}] - \nu''_2\, \psi''_0\, \varphi''_0 &= \psi''_0\, A''_2, \\ D\{\varphi''_0\, \psi''_2\} + \frac{1}{\sqrt{\mu}}\, D[\varphi''_0\, \psi''_{3/2}] - \nu''_2\, \varphi''_0\, \psi''_0 &= \varphi''_0\, B''_2. \end{aligned} \tag{10}$$

Here, we used the notations (15) and (13) of Section 4.

In the case of ordinary minor planets, we have demonstrated the relations

$$[\psi'_0\, A'_2] = [[\psi'_0\, A'_2]],$$
$$[\varphi'_0\, B'_2] = [[\varphi'_0\, B'_2]],$$
$$[\psi''_0\, A''_2] = [[\psi''_0\, A''_2]],$$
$$[\varphi''_0\, B''_2] = [[\varphi''_0\, B''_2]].$$

The proof, which is given on pp. 24 and 25 of Part I of this research, remains applicable also to the case of interest here, because of the relation (22) of Section 19 which is valid for m = 2 (in the actual case, where q = 2). /57

In addition, the functions ψ'_0, A'_2 and $\varphi'_0 B'_2$ as well as $\psi''_0 A''_2$ and $\varphi''_0 B''_2$ are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0. From this it can be concluded that

$$[[\psi'_0\, A'_2]] = [[\varphi'_0\, B'_2]],$$
$$[[\psi''_0\, A''_2]] = [[\varphi''_0\, B''_2]].$$

In view of this, eqs.(10) can be satisfied by putting

$$\nu'_2 = -\frac{[[\psi'_0\, A'_2]]}{\varphi'_0\, \psi'_0}, \qquad \nu''_2 = -\frac{[[\psi''_0\, A''_2]]}{\varphi''_0\, \psi''_0}; \tag{11}$$

$$[\psi'_0\, \varphi'_{3/2}] = [\varphi'_0\, \psi'_{3/2}] = [\psi''_0\, \varphi''_{3/2}] = [\varphi''_0\, \psi''_{3/2}] = 0 \tag{12}$$

and by giving, to the functions

$$\{\psi'_0\, \varphi'_2\}, \qquad \{\varphi'_0\, \psi'_2\}, \qquad \{\psi''_0\, \varphi''_2\}, \qquad \{\varphi''_0\, \psi''_2\} \tag{13}$$

certain well-defined expressions. The first two of these expressions as well as the two last ones are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0. The functions (13) are even polynomials of the sixth degree with respect to e_0, φ'_0, ψ'_0, φ''_0, ψ''_0. The quantities ν'_2 and ν''_2 are polynomials of the second degree in e_0^2, φ'_0, ψ'_0 and $\varphi''_0\psi''_0$, i.e., in e_0^2, $\rho_0'^2$ and $\rho_0''^2$.

Let us then compare the terms containing $\mu^{5/2}$ in the two members of eqs.(41) of Section 19. This will yield the equations

$$-D\{\psi'_0\, \varphi'_{5/2}\} - \frac{1}{\sqrt{\mu}}\, D[\psi'_0\, \varphi'_2] - \nu'_{1/2}\, \psi'_0\, \varphi'_0 = 0,$$
$$D\{\varphi'_0\, \psi'_{5/2}\} + \frac{1}{\sqrt{\mu}}\, D[\varphi'_0\, \psi'_2] - \nu'_{1/2}\, \varphi'_0\, \psi'_0 = 0,$$

$$-D\{\psi''_0\,\varphi''_{5/2}\} - \frac{1}{\sqrt{\mu}}\,D[\psi''_0\,\varphi''_2] - \nu''_{5/2}\,\psi''_0\,\varphi''_0 = 0,$$

$$D\{\varphi''_0\,\psi''_{5/2}\} + \frac{1}{\sqrt{\mu}}\,D[\varphi''_0\,\psi''_2] - \nu''_{5/2}\,\varphi''_0\,\psi''_0 = 0.$$

It follows from this that /58

$$\nu'_{5/2} = \nu''_{5/2} = 0;$$

$$[\psi'_0\,\varphi'_2] = [\varphi'_0\,\psi'_2] = [\psi''_0\,\varphi''_2] = [\varphi''_0\,\psi''_2] = 0;$$

$$\{\psi'_0\,\varphi'_{5/2}\} = \{\varphi'_0\,\psi'_{5/2}\} = \{\psi''_0\,\varphi''_{5/2}\} = \{\varphi''_0\,\psi''_{5/2}\} = 0. \tag{14}$$

The next step will furnish the quantities

$$\nu'_3, \qquad \nu''_3 \tag{15}$$

as well as the functions

$$[\psi'_0\,\varphi'_{5/2}], \quad [\varphi'_0\,\psi'_{5/2}], \quad [\psi''_0\,\varphi''_{5/2}], \quad [\varphi''_0\,\psi''_{5/2}]; \tag{16}$$

$$\{\psi'_0\,\varphi'_3\}, \quad \{\varphi'_0\,\psi'_3\}, \quad \{\psi''_0\,\varphi''_3\}, \quad \{\varphi''_0\,\psi''_3\}. \tag{17}$$

The second members of the equations [analogous to eqs.(10)], from which all these quantities are determined, are even and of the eighth degree with respect to e_0, φ'_0, ψ'_0, φ'_0, ψ''_0. It follows from this that ν'_3 and ν''_3 are polynomials of the third degree in e_0^2, $\rho_0'^2$, $\rho_0''^2$ and that the functions (16) and (17) are of the eighth degree in e_0, φ'_0, ψ'_0, φ''_0, ψ''_0.

Evidently we can continue in this manner without ever being stopped.

By induction from i to i + 1 and proceeding approximately as in the proof of theorem 1 of Section 5, we can demonstrate that the quantities

$$\nu'_{\frac{k}{2}}, \qquad \nu''_{\frac{k}{2}}, \qquad (k = 6, 7, 8, \ldots) \tag{18}$$

are polynomials of the degree k - 3 with respect to e_0^2, $\rho_0'^2$, and $\rho_0''^2$ and that the functions

$$\left[\psi'_0\,\varphi'_{\frac{k-1}{2}}\right], \quad \left[\varphi'_0\,\psi'_{\frac{k-1}{2}}\right], \quad \left[\psi''_0\,\varphi''_{\frac{k-1}{2}}\right], \quad \left[\varphi''_0\,\psi''_{\frac{k-1}{2}}\right],$$

$$(k = 6, 7, 8, \ldots) \tag{19}$$

$$\left\{\psi'_0\,\varphi'_{\frac{k}{2}}\right\}, \quad \left\{\varphi'_0\,\psi'_{\frac{k}{2}}\right\}, \quad \left\{\psi''_0\,\varphi''_{\frac{k}{2}}\right\}, \quad \left\{\varphi''_0\,\psi''_{\frac{k}{2}}\right\},$$

are polynomials of the degree 2k - 4 with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0''.

After having formed the expansions (2), it is easy to write the solution /59 of the canonical system (33) of Section 17. Let us recall that the variables φ', ψ', φ'', ψ'' and the variables ξ_1^*, η_1^*, ξ_2^*, η_2^* are linked by the formulas (12) and (30) of Section 19. By putting, in eqs.(2),

$$\mu^{\frac{k}{2}}\varphi'_{\frac{k-1}{2}} = \sum_{j',j''} A^{(k)}_{j',j''} e^{\sqrt{-1}(j'w'+j''w'')},$$
$$(k = 3, 5, 6, 7, \ldots) \qquad (20)$$
$$\mu^{\frac{k}{2}}\varphi''_{\frac{k-1}{2}} = \sum_{j',j''} B^{(k)}_{j',j''} e^{\sqrt{-1}(j'w'+j''w'')},$$

and by introducing there

$$\varepsilon' = \sqrt{\mu}\,\varrho'_0, \qquad \varepsilon'' = \sqrt{\mu}\,\varrho''_0, \qquad e' = \sqrt{\mu}\,e_0, \qquad (21)$$

the general solution of the system (33) of Section 17 (for q = 2) will assume the form

$$\begin{aligned}
\xi_1^* &= \bar{\xi} + \varepsilon' \cos w' + \sum_{k=3,5,6,\ldots}^{\infty} \sum_{j',j''} A^{(k)}_{j',j''} \cos(j'w' + j''w''),\\
\eta_1^* &= \varepsilon' \sin w' + \sum_{k=3,5,6,\ldots}^{\infty} \sum_{j',j''} A^{(k)}_{j',j''} \sin(j'w' + j''w''),\\
\xi_2^* &= \varepsilon'' \cos w'' + \sum_{k=3,5,6,\ldots}^{\infty} \sum_{j',j''} B^{(k)}_{j',j''} \cos(j'w' + j''w''),\\
\eta_2^* &= \varepsilon'' \sin w'' + \sum_{k=3,5,6,\ldots}^{\infty} \sum_{j',j''} B^{(k)}_{j',j''} \sin(j'w' + j''w'').
\end{aligned} \qquad (22)$$

The quantity $\bar{\xi}:e'$, given by eq.(10) of Section 19, can be expanded in powers of e'^2 and μ. [The coefficients of this expansion are rational in $\sqrt{\mu}$ since the coefficients of the expansion (1) of Section 19 are linear in $\sqrt{\mu}$.]

The quantities $A^{(k)}_{j',j''}$ and $B^{(k)}_{j',j''}$ are of the order of magnitude k (considering ε', ε'', e', and $\sqrt{\mu}$ as quantities of the first order of magnitude). These are /60 odd polynomials in ε', ε'', e' with coefficients that are rational in $\sqrt{\mu}$. The quantities $A^{(3)}$ and $B^{(3)}$ are of the third degree, while $A^{(5)}$ and $B^{(5)}$ are of the fifth degree with respect to ε', ε'', e'. The degree of $A^{(k)}_{j',j''}$ $(k \geq 6)$ with respect to ε', ε'', e' is 2k - 5 if $j' - j'' = +1$ and 2k - 7 if $j' - j'' \neq +1$. The degree of $B^{(k)}_{j',j''}$ $(k \geq 6)$ with respect to ε', ε'', e' is 2k - 5 provided that $j' - j'' = -1$ and 2k - 7 if $j' - j'' \neq -1$. These polynomials contain the factor $\varepsilon'^{|j'|}\varepsilon''^{|j''|}$ if $j' + j''$ is odd and the factor $e'\varepsilon'^{|j'|}\varepsilon''^{|j''|}$ if $j' + j''$ is even. The other factor is a polynomial in ε'^2, ε''^2, and e'^2. As soon as $k \geq 6$, the quantity $\sqrt{\mu}$ may appear in $A^{(k)}_{j',j''}$ and $B^{(k)}_{j',j''}$ raised to a negative power

$$(\sqrt{\mu})^{-s}.$$

In $A^{(k)}_{j',j''}$ we have $s = k - 5$ if $j' - j'' = +1$ but $s = k - 7$ if $j' - j'' \neq +1$; in $B^{(k)}_{j',j''}$ we have $s = k - 5$ if $j' - j'' = -1$ but $s = k - 7$ if $j' - j'' \neq -1$.

We still have

$$A^{(k)}_{+1,0} \equiv 0, \qquad B^{(k)}_{0,+1} \equiv 0, \qquad (k = 3, 5, 6, 7, \ldots),$$

(since the functions $\psi'_0\, \varphi'_{\frac{k-1}{2}}$, $\varphi'_0\, \psi'_{\frac{k-1}{2}}$, $\psi''_0\, \varphi''_{\frac{k-1}{2}}$, $\varphi''_0\, \psi''_{\frac{k-1}{2}}$ do not enclose any constant term).

In addition, the coefficients

$$A^{(k)}_{0,0} \qquad (k = 3, 5, 6, 7, \ldots)$$

vanish if $\epsilon' = \epsilon'' = 0$ [since the special solution which is then obtained must coincide with the particular solution (7) of Section 19].

Among the coefficients that contain $\sqrt{\mu}$ raised to a negative power, the most important ones are those of the sixth order of magnitude, namely,

$$A^{(6)}_{3,2}, \qquad A^{(6)}_{-1,-2}, \qquad B^{(6)}_{2,3}, \qquad B^{(6)}_{-2,-1}.$$

They are of the seventh degree with respect to ϵ', ϵ'', and e'.

Finally, the arguments w' and w'' have the form

$$w' = \nu' t_1 + \gamma', \qquad w'' = \nu'' t_1 + \gamma''.$$

In the expansions (2) of ν' and ν'', the quantities /61

$$\mu^{\frac{k}{2}} \nu'_{\frac{k}{2}} \text{ and } \mu^{\frac{k}{2}} \nu''_{\frac{k}{2}}$$

are of the order of magnitude k. In addition, these quantities are polynomials with respect to ϵ'^2, ϵ''^2, and e'^2, while $\mu\nu'_1$ and $\mu\nu''_1$ are of the first degree; $\mu^2\nu'_2$ and $\mu^2\nu''_2$ are of the second degree; finally, $\mu^{\frac{k}{2}} \nu'_{\frac{k}{2}}$ and $\mu^{\frac{k}{2}} \nu''_{\frac{k}{2}}$ (for k = 6, 7, 8, ...) are of the degree k - 3 in ϵ'^2, ϵ''^2, e'^2.

Let us now pass to eq.(44) of Section 19.

In the function G, we will replace φ', ψ', φ'', ψ'' by their expansions (2).

On expanding in powers of $\sqrt{\mu}$, we obtain

$$
\begin{aligned}
G &= \sum_{m=0}^{\infty} \mu^m G^{(m)} = \sum \mu^{\frac{i}{2}} g^{(\frac{i}{2})} \\
&= g^{(0)} + \mu g^{(1)} + \mu^2 g^{(2)} + \mu^{\frac{5}{2}} g^{(\frac{5}{2})} + \mu^3 g^{(3)} + \cdots .
\end{aligned} \tag{23}
$$

It is easy to give the expressions for the four first terms of the expansion (23). Let us denote $\bar{G}^{(m)}$ for $G^{(m)}$ by writing φ'_0, ψ'_0, φ''_0, ψ''_0 instead of φ', ψ', φ'', ψ''. After this, the mentioned expressions obviously will be

$$
\begin{aligned}
g^{(0)} &= \bar{G}^{(0)} = G_{0,0}^{0,0,0} + G_{1,0}^{0,1,0}(\varphi'_0 + \psi'_0) \\
&\qquad + G_{0,0}^{0,2,0} \varphi'_0 \psi'_0 + G_{0,0}^{0,0,2} \varphi''_0 \psi''_0, \\
g^{(1)} &= \bar{G}^{(1)} + G_{1,0}^{0,1,0}(\varphi'_1 + \psi'_1) + G_{0,0}^{0,2,0}(\psi'_0 \varphi'_1 + \varphi'_0 \psi'_1) \\
&\qquad + G_{0,0}^{0,0,} (\psi''_0 \varphi''_1 + \varphi''_0 \psi''_1), \\
g^{(2)} &= \bar{G}^{(2)} + \frac{d\bar{G}^{(1)}}{d\varphi'_0} \varphi'_1 + \frac{d\bar{G}^{(1)}}{d\psi'_0} \psi'_1 + \frac{d\bar{G}^{(1)}}{d\varphi''_0} \varphi''_1 + \frac{d\bar{G}^{(1)}}{d\psi''_0} \psi''_1 \\
&\qquad + G_{1,0}^{0,1,0}(\varphi'_2 + \psi'_2) + G_{0,0}^{0,2,0} \varphi'_1 \psi'_1 + G_{0,0}^{0,0,2} \varphi''_1 \psi''_1 \\
&\qquad + G_{0,0}^{0,2,0}(\psi'_0 \varphi'_2 + \varphi'_0 \psi'_2) + G_{0,0}^{0,0,2}(\psi''_0 \varphi''_2 + \varphi''_0 \psi''_2),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
g^{(\frac{5}{2})} &= G_{1,0}^{0,1,0}\left(\frac{\left[\psi'_0 \varphi'_{\frac{5}{2}}\right]}{\psi'_0} + \frac{\left[\varphi'_0 \psi'_{\frac{5}{2}}\right]}{\varphi'_0}\right) \\
&\quad + G_{0,0}^{0,2,0}\left(\left[\psi'_0 \varphi'_{\frac{5}{2}}\right] + \left[\varphi'_0 \psi'_{\frac{5}{2}}\right]\right) \\
&\quad + G_{0,0}^{0,0,2}\left(\left[\psi''_0 \varphi''_{\frac{5}{2}}\right] + \left[\varphi''_0 \psi''_{\frac{5}{2}}\right]\right).
\end{aligned}
$$ /62

As in Section 20, we can see that $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$ are polynomials of the form

$$
\sum \varphi_0'^{\alpha'} \psi_0'^{\beta'} \varphi_0''^{\alpha''} \psi_0''^{\beta''},
$$

in which the exponents α', β', α'', β'' satisfy, respectively, the conditions (8), (9), and (10) of Section 20. From this, we can conclude that $[g^{(0)}]$, $[g^{(1)}]$, and $[g^{(2)}]$ are constants.

It is easy to demonstrate that $[g^{(5/2)}]$ is of the order of $\sqrt{\mu}$. In fact, the quantity $G_{0,0}^{0,2,0} + G_{0,0}^{0,0,2}$ is of the order of $\sqrt{\mu}$ for q = 2, whereas the function $\left[\psi'_0 \varphi'_{\frac{5}{2}} + \varphi'_0 \psi'_{\frac{5}{2}} - \psi''_0 \varphi''_{\frac{5}{2}} - \varphi''_0 \psi''_{\frac{5}{2}}\right]$ is also of the order of $\sqrt{\mu}$, because of the first integral $H = h$.

It is obvious that $g^{(i/2)}$ is an even polynomial in φ_0, φ'_0, ψ'_0, φ''_0, ψ''_0. We state that this polynomial is of the degree $2i - 2$ as long as $i > 4$. We also state that the terms of the degree $2i - 2$ of $g^{(i/2)}$ are terms of the degree $2i - 2$ of the function

$$
\begin{aligned}
&G^{0,1,0}_{1,0}\left(\frac{\left[\psi'_0\,\varphi'_{\frac{i}{2}}\right]}{\psi'_0} + \frac{\left[\varphi'_0\,\psi'_{\frac{i}{2}}\right]}{\varphi'_0}\right)\\
&+ G^{0,2,0}_{0,0}\left(\left[\psi'_0\,\varphi'_{\frac{i}{2}}\right] + \left[\varphi'_0\,\psi'_{\frac{i}{2}}\right]\right)\\
&+ G^{0,0,2}_{0,0}\left(\left[\psi''_0\,\varphi''_{\frac{i}{2}}\right] + \left[\varphi''_0\,\psi''_{\frac{i}{2}}\right]\right).
\end{aligned}
\tag{25}
$$

In fact, the function $g^{(i/2)}$ is composed of terms having the form

/63

$$
\begin{aligned}
&C\varphi_1'^{m'_1}\psi_1'^{n'_1}\varphi_1''^{m''_1}\psi_1''^{n''_1}\varphi_2'^{m'_2}\psi_2'^{n'_2}\varphi_2''^{m''_2}\psi_2''^{n''_2}\varphi_{\frac{5}{2}}'^{m'_{\frac{5}{2}}}\psi_{\frac{5}{2}}'^{n'_{\frac{5}{2}}}\varphi_{\frac{5}{2}}''^{m''_{\frac{5}{2}}}\psi_{\frac{5}{2}}''^{n''_{\frac{5}{2}}}\ldots\\
&\ldots\varphi_{\frac{i}{2}}'^{m'_{\frac{i}{2}}}\psi_{\frac{i}{2}}'^{n'_{\frac{i}{2}}}\varphi_{\frac{i}{2}}''^{m''_{\frac{i}{2}}}\psi_{\frac{i}{2}}''^{n''_{\frac{i}{2}}}\frac{d^{\Sigma(m'+n'+m''+n'')}G^{(m)}}{(d\varphi'_0)^{\Sigma m'}(d\psi'_0)^{\Sigma n'}(d\varphi''_0)^{\Sigma m''}(d\psi''_0)^{\Sigma n''}},
\end{aligned}
\tag{26}
$$

where the exponents satisfy the condition

$$
\sum_{k=3,5,6,\ldots i+1}\frac{k-1}{2}\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right) + m = \frac{i}{2}.
\tag{27}
$$

According to the above statements (pp 145-146), the functions $\psi'_{\frac{k-1}{2}}$, $\varphi'_{\frac{k-1}{2}}$, $\varphi''_{\frac{k-1}{2}}$, $\psi''_{\frac{k-1}{2}}$ are of the degree $2k - 5$ as long as $k = 5, 6 \ldots$.

Thus, the polynomial (26) with respect to φ_0, φ'_0, ψ'_0, φ''_0, ψ''_0, is of the degree

$$
\begin{aligned}
I &= 3(m'_1 + n'_1 + m''_1 + n''_1)\\
&+ \sum_{k=5}^{i+1}(2k-5)\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right)\\
&+ 2m + 2 - \sum(m' + n' + m'' + n'')\\
&= 2(m'_1 + n'_1 + m''_1 + n''_1) + 2m + 2\\
&+ \sum_{k=3,5,6,\ldots i+1}(2k-6)\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right).
\end{aligned}
$$

In view of the condition (27), we have

$$I = 2i + 2 - 2m - 2(m'_1 + n'_1 + m''_1 + n''_1)$$
$$- 4\sum_{k=5}^{i+1}\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right).$$

Consequently, we always have $I \leq 2i + 2$.

It is possible to have $I = 2i + 2$ only if

$$m = 0, \qquad \sum (m' + n' + m'' + n'') = 0$$

i.e., because of eq.(27), only for $i/2 = 0$.

It is possible to have $I = 2i$ only if /64

$$m + (m'_1 + n'_1 + m''_1 + n''_1) = 1,$$
$$\sum_{k=5}^{i+1}\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right) = 0$$

i.e., because of eq.(27), only for $i/2 = 1$.

It is possible to have $I = 2i - 2$ only in one of the two cases (α) and (β):

(α)
$$m + (m'_1 + n'_1 + m''_1 + n''_1) = 2,$$
$$\sum_{k=5}^{i+1}\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right) = 0;$$

(β)
$$m + (m'_1 + n'_1 + m''_1 + n''_1) = 0,$$
$$\sum_{k=5}^{i+1}\left(m'_{\frac{k-1}{2}} + n'_{\frac{k-1}{2}} + m''_{\frac{k-1}{2}} + n''_{\frac{k-1}{2}}\right) = 1.$$

The case (α) can be realized only if $i/2 = 2$. In the case (β), the function (26) reduces to one or another of the four functions

$$\varphi'_i \frac{d\bar{G}^{(0)}}{d\varphi'_0}, \qquad \psi'_i \frac{d\bar{G}^{(0)}}{d\psi'_0}, \qquad \varphi''_i \frac{d\bar{G}^{(0)}}{d\varphi''_1}, \qquad \psi''_i \frac{d\bar{G}^{(0)}}{d\psi''_0}.$$

It follows from this that the part of $g^{(i/2)}$, which is of the degree $2i - 2$ (if

$i > 4$), is present in the function which is the sum of these four functions or else in the function (25).

Finally, because of the first integral $H = h$, the terms of the degree $2i - 2$ in the function $[g^{(i/2)}]$ obviously contain $\sqrt{\mu}$ as factor.

Now, we can write the various functions that appear in the expansion (23) in the form of

$$g^{(\frac{i}{2})} = \sum_{j', j''} g^{(\frac{i}{2})}_{j', j''} e^{\sqrt{-1}(j'w' + j''w'')}. \tag{28}$$

According to the above statements, the quantity $g^{(i/2)}_{j', j''}$ is an even polynomial /65 in ρ'_0, ρ''_0, and e_0. The degree of $g^{(0)}_{j', j''}$ is 2; the degree of $g^{(1)}_{j', j''}$ is 4; the degree of $g^{(2)}_{j', j''}$ is 6; the degree of $g^{(i/2)}_{j', j''}$ (for $i > 4$) is $2i - 2$ if $|j' - j''| = 0$ or 1, and $2i - 4$ if $|j' - j''| > 1$. Finally, the part of the degree $2i - 2$ of the function $g^{(i/2)}_{j', j''}$ includes $\sqrt{\mu}$ as factor.

In view of the expansion (23) as well as of the expression (28), eq.(44) of Section 19, after integration, will yield

$$y_1^* = nt + c - \sum_{j', j''} \sum_{i=0,2,4,5\ldots}^{\infty} \frac{\mu^{\frac{i}{2}+1} g^{(\frac{i}{2})}_{j', j''}}{j'\nu' + j''\nu''} \sin(j'w' + j''w''). \tag{29}$$

The integer j'' is even, so that we have $j' j'' \neq 0$. The quantity c is an arbitrary constant. The expression of the constant n will be given later in the text.

The quantities ν' and ν'' are expanded in powers of $\sqrt{\mu}$ by the two last of the equations in the system (2). We will expand the quantity $(j'\nu' + j''\nu'')^{-1}$ in powers of $\sqrt{\mu}$. Then, two cases must be differentiated depending on whether $j' = j''$ or $j' \neq j''$.

We will assume first that

$$j' = j'' = j.$$

By putting, for abbreviation,

$$\sqrt{\mu}\, \frac{\nu'_k + \nu''_k}{\nu'_0 + \nu''_0} = \nu_{\frac{k}{2}}, \qquad (k = 2, 4, 6, 7, \ldots),$$

we can set

$$\frac{1}{j(\nu' + \nu'')} = \frac{1}{\sqrt{\mu}} \frac{\sqrt{\mu}}{j(\nu'_0 + \nu''_0)} \Big\{ 1 + \mu^{\frac{1}{2}} \nu_1 + \mu^{\frac{3}{2}} \nu_2 + \mu^{\frac{5}{2}} \nu_3 + \mu^3 \nu_{\frac{7}{2}} + \cdots \Big\}^{-1} \tag{30}$$

$$= \frac{1}{\sqrt{\mu}} \left\{ \delta_j^{(0)} + \mu^{\frac{1}{2}} \delta_j^{(\frac{1}{2})} + \mu \delta_j^{(1)} + \mu^{\frac{3}{2}} \delta_j^{(\frac{3}{2})} + \cdots \right\}.$$

We will then search for the degree of the polynomial $\delta_j^{(s/2)}$. This function is composed of terms of the form of /66

$$C \nu_1^{n_1} \nu_2^{n_2} \nu_3^{n_3} \nu_{\frac{7}{2}}^{n_{\frac{7}{2}}} \cdots \nu_{\frac{s+1}{2}}^{n_{\frac{s+1}{2}}}. \tag{31}$$

where C is a numerical constant. The exponents $n_1, n_2, \ldots n_{\frac{s+1}{2}}$ satisfy the condition

$$\sum_{k=2,4,6,7\ldots}^{s+1} (k-1)\, n_{\frac{k}{2}} = s.$$

We know that ν_1 and ν_2 are of the degrees 2 resp. 4 and that $\nu_{k/2}$ is of the degree $2k - 6$ for $k = 6, 7, \ldots$. The degree of the term (31) consequently will be

$$N = 2n_1 + 4n_2 + \sum_{k=6}^{s+1} (2k-6)\, n_{\frac{k}{2}} = 2s - 2n_2 - 4 \sum_{k=6}^{s+1} n_{\frac{k}{2}}.$$

We therefore have

$$N \leq 2s$$

and

$$N = 2s$$

by putting, in eq.(31),

$$n_1 = s, \qquad n_2 = n_3 = \cdots = n_{\frac{s+1}{2}} = 0.$$

Consequently, we see that $\delta_j^{(s/2)}$ is of the degree 2s with respect to ρ_0', ρ_0'', and e_0.

Now, we will put

$$g_{j,j}^{(\frac{i}{2})} = g_{j,j}^{\prime(\frac{i}{2})} + \sqrt{\mu}\, g_{j,j}^{\prime\prime(\frac{i}{2})},$$

where the first polynomial is of the degree $2i - 4$ and the second is of the degree $2i - 2$, with respect to ρ_0', ρ_0'', and e_0.

We know that /67

$$g'^{(\frac{5}{2})}_{j,j} = 0.$$

By making use of the expansion (30), we can introduce into eq.(29)

$$-\sum_{i=5}^{\infty} \frac{\mu^{\frac{i}{2}+1} g^{(\frac{i}{2})}_{j,j}}{j(\nu' + \nu'')} = \sum_{k=7}^{\infty} C^{(k)}_{j,j} \tag{32}$$

by putting

$$C^{(k)}_{j,j} = -\mu^{\frac{k}{2}} \sum_{s=0}^{k-7} \delta^{(\frac{s}{2})}_{j} \left(g'^{(\frac{k-s-1}{2})}_{j,j} + g''^{(\frac{k-s-2}{2})}_{j,j} \right). \quad (k=7,8,\ldots) \tag{33}$$

Obviously, the quantity $C^{(k)}_{j,j}$ is of the order of magnitude k. According to what we know of the polynomials $\delta^{(s/2)}_j$, $g'^{(\frac{k-s-1}{2})}_{j,j}$ and $g''^{(\frac{k-s-2}{2})}_{j,j}$, we can also conclude that $C^{(k)}_{j,j}$ is an even polynomial of the degree 2k - 6 with respect to ρ'_0, ρ''_0, and e_0 or, if preferable, with respect to ϵ', ϵ'', and e'.

We will now assume that

$$j' \neq j''.$$

Then, by putting for abbreviation

$$\frac{j' \nu'_{\frac{k}{2}} + j'' \nu''_{\frac{k}{2}}}{j' \nu'_0 + j'' \nu''_0} = \nu_{\frac{k}{2}},$$

we obtain the expansion

$$\begin{aligned}\frac{1}{j'\nu' + j''\nu''} &= \frac{1}{j'\nu'_0 + j''_0\nu''_0} \Big\{1 + \mu\nu_1 + \mu^2\nu_2 \\ &\qquad + \mu^3\nu_3 + \mu^{\frac{7}{2}}\nu_{\frac{7}{2}} + \cdots\Big\}^{-1} \\ &= \delta^{(0)}_{j',j''} + \mu\,\delta^{(1)}_{j',j''} + \mu^2\delta^{(2)}_{j',j''} + \mu^3\delta^{(3)}_{j',j''} + \mu^{\frac{7}{2}}\delta^{(\frac{7}{2})}_{j',j''} + \cdots .\end{aligned} \tag{34}$$

Let us then search for the polynomial $\delta^{(s/2)}_{j',j''}$ with respect to ρ'_0, ρ''_0, and e_0. This polynomial is composed of several parts, having the form /68

$$C\, \nu_1^{n_1} \nu_2^{n_2} \nu_3^{n_3} \nu_{\frac{7}{2}}^{n_{\frac{7}{2}}} \ldots \nu_{\frac{s}{2}}^{n_{\frac{s}{2}}}, \tag{35}$$

where C is a numerical constant; the integers n_1, n_2, ... $n_{s/2}$ satisfy the relation·

$$\sum_{k=2,4,6,7,\ldots}^{s} k\, n_{\frac{k}{2}} = s.$$

The part (35) obviously is of the degree

$$N = 2n_1 + 4n_2 + \sum_{k=6}^{s} (2k-6)\, n_{\frac{k}{2}} = 2s - 2n_1 - 4n_2 - 6\sum_{k=6}^{s} n_{\frac{k}{2}}.$$

It is possible to have:

$N = 2s$, only if $n_1 = n_2 = \cdots = n_{\frac{s}{2}} = 0,$

i.e., for s = 0;

$N = 2s-2,$ " " $n_1 = 1,\ n_2 = n_3 = \cdots = n_{\frac{s}{2}} = 0,$

i.e., for s = 2;

$N = 2s-4,$ " " $2n_1 + 4n_2 = 4,\ n_3 = n_4 = \cdots = n_{\frac{s}{2}} = 0,$

i.e., for s = 4.

For the other values of s, we will have

$$N = 2s-6 \quad \text{by putting} \quad n_1 = n_2 = \cdots = n_{\frac{s-1}{2}} = 0, \quad n_{\frac{s}{2}} = 1.$$

Consequently, in eq.(34), the quantity $\delta^{(0)}_{j',j''}$ is of the degree 0, while $\delta^{(1)}_{j',j''}$ is of the degree 2, $\delta^{(2)}_{j',j''}$ is of the degree 4, and $\delta^{(s/2)}_{j',j''}$ for $s \geq 6$ is of the degree $2s - 6$ with respect to ρ_0', ρ_0'', and e_0.

We will now put, in eq.(29) and making use of the expansion (34), /69

$$-\sum_{i=0,2,4,5\ldots}^{\infty} \frac{\mu^{\frac{i}{2}+1} g^{(\frac{i}{2})}_{j',j''}}{j'\nu' + j''\nu''} = \sum_{k=2,4,6,7,\ldots}^{\infty} C^{(k)}_{j',j''} \qquad (j' \neq j''). \tag{36}$$

Here, we have used the notation

$$C^{(k)}_{j',j''} = -\mu^{\frac{k}{2}} \sum_{s=0}^{k-2} \delta^{(\frac{s}{2})}_{j',j''}\, g^{(\frac{k-s-2}{2})}_{j',j''}, \tag{37}$$

$$(k = 2, 4, 6, 7, \ldots), \qquad (j' \neq j'').$$

This quantity is of the order of magnitude k; in addition this is an even polynomial with respect to ρ_0', ρ_0'', and e or else with respect to ϵ', ϵ'', and e'. The quantity $C^{(2)}_{j',j''}$ is of the second degree; $C^{(4)}_{j',j''}$ is of the fourth degree; $C^{(6)}_{j',j''}$ is of the sixth degree; finally, for k = 7, 8, ... , the polynomial $C^{(k)}_{j',j''}$ is of the degree 2k - 6 if $|j' - j''| = 1$ and of the degree 2k - 8 if $|j' - j''| > 1$.

Now, in view of eqs.(32), (33), (36), and (37), the expression (29) takes the definite form

$$y_1^* = nt + c + \sum_{k=2,4,6,7,\ldots}^{\infty} \sum_{j',j''} C^{(k)}_{j',j''} \sin(j'w' + j''w''). \tag{38}$$

This is a formula analogous to eqs.(22).

The quantities $C^{(k)}_{j',j''}$ are of the order of magnitude k (while considering ϵ', ϵ'', e', and $\sqrt{\mu}$ as being of the first order of magnitude). These quantities are even polynomials in ϵ', ϵ'', e' with coefficients that are rational with respect to $\sqrt{\mu}$. The polynomials $C^{(2)}_{j',j''}$ are of the degree 2; $C^{(4)}_{j',j''}$ of the degree 4; $C^{(6)}_{j',j''}$ of the degree 6 with respect to ϵ', ϵ'', e'. The degree of $C^{(k)}_{j',j''}$ ($k \geq 6$) with respect to ϵ', ϵ'', and e' is 2k - 6 if $|j' - j''| = 0$ or 1 but only 2k - 8 if $|j' - j''| > 1$. All these polynomials contain the factor $\epsilon'^{|j'|}\epsilon''^{|j''|}$, if j' is even and the factor $e'\epsilon'^{|j'|}\epsilon''^{|j''|}$ if j' is odd. The other factor is a polynomial in ϵ'^2, ϵ''^2, and e'^2.

As soon as $k \geq 7$, the quantity $\sqrt{\mu}$ may appear in $C^{(k)}_{j',j''}$, raised to a negative power /70

$$(\sqrt{\mu})^{-s}.$$

According to the above statements, we have s = k - 6 if $|j' - j''| = 0$ or 1 and s = k - 8 if $|j' - j''| > 1$. Among the polynomials $C^{(k)}_{j',j''}$ that contain $\sqrt{\mu}$ raised to a negative power, the most important are

$$C^{(7)}_{3,2}, \quad C^{(7)}_{1,2}, \quad C^{(7)}_{2,2}$$

(as well as the three $C^{(7)}_{-3,-2}$, $C^{(7)}_{-1,-2}$, $C^{(7)}_{-2,-2}$ which are identical to these). The first two polynomials originate in the function $g^{(5/2)} - [g^{(5/2)}]$ [see eq.(24)]. The last polynomial has its origin in the functions $[g^{(5/2)}]$ and $[g^{(3)}]$.

The absolute mean motion can be expanded in the form of

$$n = n^{(0)} + n^{(2)} + n^{(4)} + n^{(6)} + \cdots.$$

The quantity $n^{(k)}$ (k = 4, 6, 8, 10, 11, 12, ...) which has the expression

$$n^{(k)} = -\mu^{\frac{k}{2}}[[g^{(\frac{k}{2}-2)}]], \qquad (k = 4, 6, 8, 10, 11, 12, \ldots),$$

is of the order of magnitude k. The quantities $n^{(4)}$, $n^{(6)}$, $n^{(8)}$ are polynomials of the degree 1, 2, resp. 3 with respect to ϵ'^2, ϵ''^2, and e'^2. For $k \geq 10$, $n^{(k)}$ is a polynomial of the degree $k - 6$ with respect to ϵ'^2, ϵ''^2, and e'^2. We have, specifically,

$$n^{(0)} = x_1^{*-3}, \qquad n^{(2)} = -\mu \frac{d\,\mathfrak{F}^{1,0,0,0}_{0,0}}{d x_1^*},$$

$$\begin{aligned} n^{(4)} &= -\mu^2 [[g^{(0)}]] = -\mu^2 (G^{0,0,0}_{0,0} + G^{0,2,0}_{0,0} \varrho_0'^2 + G^{0,0,2}_{0,0} \varrho_0''^2) \\ &= -\mu^2 \frac{d\,\mathfrak{F}^{2,0,0,0}_{0,0}}{d x_1^*} - \mu \varepsilon'^2 \frac{d\,\mathfrak{F}^{1,0,2,0}_{0,0}}{d x_1^*} - \mu \varepsilon''^2 \frac{d\,\mathfrak{F}^{1,0,0,2}_{0,0}}{d x_1^*} \\ &\quad - \mu e'^2 \left\{ \frac{d\,\mathfrak{F}^{1,2,0,0}_{0,0}}{d x_1^*} - 2 \frac{d\,\mathfrak{F}^{1,1,1,0}_{1,0}}{d x_1^*} \frac{\mathfrak{F}^{1,1,1,0}_{1,0}}{\mathfrak{F}^{1,0,2,0}_{0,0}} + \frac{d\,\mathfrak{F}^{1,0,2,0}_{0,0}}{d x_1^*} \left(\frac{\mathfrak{F}^{1,1,1,0}_{1,0}}{\mathfrak{F}^{1,0,2,0}_{0,0}} \right)^2 \right\}. \end{aligned}$$

Section 22. /71

Now, we will study the secular variations in the case in which

$$q = 1.$$

In this case, the quantity $\nu_0' + \nu_0''$ is comparable in magnitude to unity.

The quantities ν_0' and ν_0'' depend on the parameter x_1^* (or else, if preferred, on the mean absolute motion of the minor planet). By varying this parameter, it may happen that a divisor

$$j' \nu'_0 + j'' \nu''_0 \tag{1}$$

becomes small (of the order of μ or smaller).

In this Section, we will assume that the divisors (1) are not small as long as the integers j', and j'' are not both zero.

We will again start from eqs.(41) of Section 19. In the actual case, we can there introduce expansions of the form

$$\begin{aligned} \varphi' &= \varphi'_0 + \mu \varphi'_1 + \mu^2 \varphi'_2 + \cdots, \\ \psi' &= \psi'_0 + \mu \psi'_1 + \mu^2 \psi'_2 + \cdots, \\ \varphi'' &= \varphi''_0 + \mu \varphi''_1 + \mu^2 \varphi''_2 + \cdots, \\ \psi'' &= \psi''_0 + \mu \psi''_1 + \mu^2 \psi''_2 + \cdots, \\ \nu' &= \nu'_0 + \mu \nu'_1 + \mu^2 \nu'_2 + \cdots, \\ \nu'' &= \nu''_0 + \mu \nu''_1 + \mu^2 \nu''_2 + \cdots \end{aligned} \tag{2}$$

On expanding the two members of the mentioned equations in powers of μ and comparing the coefficients of the same powers, we find equations that successively determine the terms of the expansions (2).

First, the equations are satisfied for $\mu = 0$.

To have the terms in μ vanish from the considered equations, it is necessary that /72

$$
\begin{aligned}
-D(\psi'_0 \varphi'_1) - \nu'_1 \psi'_0 \varphi'_0 &= 2\psi'_0 \frac{dH^{(1)}}{d\psi'_0}, \\
D(\varphi'_0 \psi'_1) - \nu'_1 \varphi'_0 \psi'_0 &= 2\varphi'_0 \frac{dH^{(1)}}{d\varphi'_0}, \\
-D(\psi''_0 \varphi''_1) - \nu''_1 \psi''_0 \varphi''_0 &= 2\psi''_0 \frac{dH^{(1)}}{d\psi''_0}, \\
D(\varphi''_0 \psi''_1) - \nu''_1 \varphi''_0 \psi''_0 &= 2\varphi''_0 \frac{dH^{(1)}}{d\varphi''_0}.
\end{aligned}
\tag{3}
$$

As in the preceding Section, we have put

$$D = \nu'_0 D' + \nu''_0 D''.$$

In addition, we will denote by

$$[[f]]$$

the mean value of any function f, which can be expanded in multiples of the arguments w' and w''. Besides this, we will denote by $\mathfrak{M}$ the monomial

$$\mathfrak{M} = \varphi_0'^{\alpha'} \psi_0'^{\beta'} \varphi_0''^{\alpha''} \psi_0''^{\beta''} = \varrho_0'^{m'} \varrho_0''^{m''} e^{\sqrt{-1}(j'w' + j''w'')}.$$

In view of this, eqs.(3) are satisfied by the expressions

$$
\begin{aligned}
\nu'_1 &= -\frac{2d[[H^{(1)}]]}{\varphi'_0 d\psi'_0} = -4H_{0,0}^{1,4,0} \varphi'_0 \psi'_0 - 2H_{0,0}^{1,2,2} \varphi''_0 \psi''_0 - 2H_{0,0}^{1,2,0}, \\
\nu''_1 &= -\frac{2d[[H^{(1)}]]}{\varphi''_0 d\psi''_0} = -4H_{0,0}^{1,0,4} \varphi''_0 \psi''_0 - 2H_{0,0}^{1,2,2} \varphi'_0 \psi'_0 - 2H_{0,0}^{1,0,2};
\end{aligned}
\tag{4}
$$

$$\psi'_0 \varphi'_1 = -\sum \frac{2\beta' H_{j',j''}^{1,m',m''}}{j'\nu'_0 + j''\nu''_0} \mathfrak{M}, \qquad /73$$

$$\varphi'_0 \psi'_1 = \sum \frac{2\alpha' H^{1,m',m''}_{j',j''}}{j'\nu'_0 + j''\nu''_0} \mathfrak{M}, \qquad (j' \neq j'')$$

$$\psi''_0 \varphi''_1 = -\sum \frac{2\beta'' H^{1,m',m''}_{j',j''}}{j'\nu'_0 + j''\nu''_0} \mathfrak{M},$$

$$\varphi''_0 \psi''_1 = \sum \frac{2\alpha'' H^{1,m',m''}_{j',j''}}{j'\nu'_0 + j''\nu''_0} \mathfrak{M}. \tag{5}$$

The integration constants have been equated to zero, which is permissible without interfering with the generality.

The functions (5) are even polynomials and of the fourth degree with respect to e_0, φ'_0, ψ'_0, φ''_0, ψ''_0. The two first as well as the two last functions are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0.

By mutually equating the coefficients of μ^2 in the expansions of the two members of eqs.(41) of Section 19, we will then obtain

$$\begin{aligned}
-D(\psi'_0 \varphi'_2) - \nu'_2 \psi'_0 \varphi'_0 &= \psi'_0 A'_2, \\
D(\varphi'_0 \psi'_2) - \nu'_2 \varphi'_0 \psi'_0 &= \varphi'_0 B'_2, \\
-D(\psi''_0 \varphi''_2) - \nu''_2 \psi''_0 \varphi''_0 &= \psi''_0 A''_2, \\
D(\varphi''_0 \psi''_2) - \nu''_2 \varphi''_0 \psi''_0 &= \varphi''_0 B''_2.
\end{aligned} \tag{6}$$

The second members are known polynomials which are even and of the sixth degree with respect to e_0, φ'_0, ψ'_0, φ''_0, ψ''_0. The functions $\psi'_0 A'_2$ and $\varphi'_0 B'_2$ as well as $\psi''_0 A''_2$ and $\varphi''_0 B''_2$ are permuted on permuting φ'_0 and ψ'_0 as well as φ''_0 and ψ''_0. Consequently, we can cause the constant terms of eqs.(6) to vanish by posing

$$\begin{aligned}
\nu'_2 &= -\frac{[[\psi'_0 A'_2]]}{\psi'_0 \varphi'_0} = -\frac{[[\varphi'_0 B'_2]]}{\varphi'_0 \psi'_0}, \\
\nu''_2 &= -\frac{[[\psi''_0 A''_2]]}{\psi''_0 \varphi''_0} = -\frac{[[\varphi''_0 B''_2]]}{\varphi''_0 \psi''_0}.
\end{aligned} \tag{7}$$

Then, these equations, after quadrature, will yield the expressions for the functions /74

$$\psi'_0 \varphi'_2, \quad \varphi'_0 \psi'_2, \quad \psi''_0 \varphi''_2, \quad \varphi''_0 \psi''_2. \tag{8}$$

The quantities (7) are polynomials of the second degree in e_0^2, $\rho_0'^2$, and $\rho_0''^2$. The functions (8) are even polynomials and of the sixth degree with respect to e_0, φ'_0, ψ'_0, φ''_0, ψ''_0. The first two functions as well as the two last functions are permuted on permuting φ_0' and ψ_0' as well as φ_0'' and ψ_0''.

Evidently, we can continue in this manner as far as desired, and successively determine the terms of the expansions (2).

It is easy to demonstrate that the quantities

$$\nu_k' \text{ and } \nu_k'' \tag{9}$$

are polynomials of the degree k in e_0^2, $\rho_0'^2$, and $\rho_0''^2$ and that the functions

$$\varphi_k', \quad \psi_k', \quad \varphi_k'', \quad \psi_k'' \tag{10}$$

are odd polynomials of the degree 2k + 1 with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0''. This theorem is proved in almost the same manner as theorem 1 of Section 5. Consequently, it seems unnecessary to discuss this further.

After integration of eqs.(41) of Section 19, it is easy to write the general solution of the canonical system (33) of Section 17, which is equivalent to these.

For this purpose, we introduce in the expansions (2)

$$\varepsilon' = \sqrt{\mu}\,\varrho'_0, \qquad \varepsilon'' = \sqrt{\mu}\,\varrho''_0, \qquad e' = \sqrt{\mu}\,e_0. \tag{11}$$

At the same time, we write there

$$\begin{aligned} \mu^{k+\frac{1}{2}}\varphi_k' &= \sum_{j',j''} A_{j',j''}^{(2k+1)} e^{\sqrt{-1}(j'w'+j''w'')}, \\ & \qquad\qquad\qquad (k = 1, 2, 3, \ldots) \\ \mu^{k+\frac{1}{2}}\varphi_k'' &= \sum_{j',j''} B_{j',j''}^{(2k+1)} e^{\sqrt{-1}(j'w'+j''w'')}. \end{aligned}$$

In view of the transformations (12) and (30) of Section 19, the wanted solution takes the form /75

$$\begin{aligned} \xi_1^* &= \xi + \varepsilon'\cos w' + \sum_{k=1}^{\infty}\sum_{j',j''} A_{j',j''}^{(2k+1)}\cos(j'w' + j''w''), \\ \eta_1^* &= \varepsilon'\sin w' + \sum_{k=1}^{\infty}\sum_{j',j''} A_{j',j''}^{(2k+1)}\sin(j'w' + j''w''), \\ \xi_2^* &= \varepsilon''\cos w'' + \sum_{k=1}^{\infty}\sum_{j',j''} B_{j',j''}^{(2k+1)}\cos(j'w' + j''w''), \\ \eta_2^* &= \varepsilon''\sin w'' + \sum_{k=1}^{\infty}\sum_{j',j''} B_{j',j''}^{(2k+1)}\sin(j'w' + j''w''). \end{aligned} \tag{12}$$

The quantity $\bar{\xi}:e'$, given by eq.(10) in Section 19, can be expanded in powers of μ and e'^2 (the coefficients of the expansion are rational in $\sqrt{\mu}$).

The quantities $A_{j',j''}^{(2k+1)}$ and $B_{j',j''}^{(2k+1)}$ are of the order 2k + 1, considering ϵ', ϵ'', e', and $\sqrt{\mu}$ as quantities of the first order of magnitude. These quantities, in addition, are odd polynomials of the degree 2k + 1 with respect to ϵ', ϵ'', and e'. These polynomials contain the factor $\epsilon'^{|j'|}\epsilon''^{|j''|}$ if $j' + j''$ is odd and the factor $e'\epsilon'^{|j'|}\epsilon''^{|j''|}$ if $j' + j''$ is even. The other factor is a polynomial in ϵ'^2, ϵ''^2, and e'^2.

We still have

$$A_{+1,0}^{(2k+1)} \equiv 0, \qquad B_{0,+1}^{(2k+1)} \equiv 0, \qquad (k = 1, 2, 3, \ldots),$$

(since the functions $\psi_0'\varphi_k'$ and $\psi_0''\varphi_k''$ include no constant term).

In addition, the coefficients

$$A_{0,0}^{(2k+1)} \qquad (k = 1, 2, 3, \ldots)$$

are canceled out if $\epsilon' = \epsilon'' = 0$ [since the special solution obtained in this case coincides with the particular solution (7) of Section 19].

Finally, the arguments w' and w'' have the form

$$w' = \nu' t_1 + \gamma', \qquad w'' = \nu'' t_1 + \gamma''.$$

In the expansions (2) of ν' and ν'', the quantities $\mu^k\nu_k'$ and $\mu^k\nu_k''$ are of the order of magnitude 2k. In addition, these quantities are polynomials of the degree k with respect to ϵ'^2, ϵ''^2, and e'^2. /76

We will next investigate the secular inequalities of the longitude.

The function G which appears in the second member of eq.(44) of Section 19, is defined by the formulas (42) and (43) of Section 19. We will there replace φ', ψ', φ'', ψ'' by their expansions (2). This will yield

$$G = \sum_{m=0}^{\infty} \mu^m G^{(m)} = \sum_{i=0}^{\infty} \mu^i g^{(i)}. \tag{13}$$

In view of the fact that the polynomial $G^{(m)}$ is even and of the degree 2m + 2 with respect to e_0, φ', ψ', φ'', ψ'' and that the polynomials (10) are odd and of the degree 2k + 1 with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0'', it is easy to demonstrate (by making use of the method given in Section 5) that $g^{(i)}$ is an even polynomial of the degree 2i + 2 with respect to e_0, φ_0', ψ_0', φ_0'', ψ_0''.

We have, specifically,

$$g^{(0)} = G_{0,0}^{0,0,0} + G_{1,0}^{0,1,0}(\varphi'_0 + \psi'_0) + G_{0,0}^{0,2,0}\varphi'_0\psi'_0 + G_{0,0}^{0,0,2}\varphi''_0\psi''_0. \qquad (14)$$

By arranging it according to multiples of the arguments w' and w'', the function $g^{(i)}$ can be given the form of

$$g^{(i)} = \sum_{j',j''} g_{j',j''}^{(i)} e^{\sqrt{-1}(j'w'+j''w'')}, \quad (i = 0, 1, 2, \ldots). \qquad (15)$$

The quantities $g_{j',j''}^{(i)}$ are even polynomials of the degree 2i + 2 with respect to ρ'_0, ρ''_0, and e_0.

Thus, eq.(44) of Section 19 will yield, after integration,

$$y_1^* = nt + c - \sum_{j',j''}\sum_{i=0}^{\infty} \frac{\mu^{i+1} g_{j',j''}^{(i)}}{j'\nu' + j''\nu''} \sin(j'w' + j''w''). \qquad (16)$$

Here, c is an arbitrary constant; n is the constant term of the second member of eq.(44) of Section 19. In the sum of the expression (16), the integers j' and j'' are not simultaneously zero.

The quantity $(j'\nu' + j''\nu'')^{-1}$ can be expanded in powers of μ because of the expansions (2) of ν' and ν''. By putting /77

$$(j'\nu' + j''\nu'')^{-1} = \delta^{(0)} + \mu\delta^{(1)} + \mu^2\delta^{(2)} + \cdots, \qquad (17)$$

it becomes obvious that $\delta^{(s)}$ is a polynomial of the degree s with respect to $\rho_0'^2$, $\rho_0''^2$, and e_0^2.

Making use of this expansion, we introduce the following expression in eq.(16):

$$-\sum_{i=0}^{\infty} \frac{\mu^{i+1} g_{j',j''}^{(i)}}{j'\nu' + j''\nu''} = \sum_{k=1}^{\infty} C_{j',j''}^{(2k)}.$$

We have here used the notation

$$C_{j',j''}^{(2k)} = -\mu^k \sum_{s=0}^{k-1} \delta^{(s)} g_{j',j''}^{(k-s-1)}, \quad (k = 1, 2, 3, \ldots). \qquad (18)$$

The solution of eq.(44) of Section 19 will thus become

$$y_1^* = nt + c + \sum_{k=1}^{\infty}\sum_{j',j''} C_{j',j''}^{(2k)} \sin(j'w' + j''w''). \qquad (19)$$

The quantities $C_{j',j''}^{(2k)}$ are of the order of magnitude 2k, considering ϵ', ϵ'', e', and $\sqrt{\mu}$ as being of the first order of magnitude. These are even polynomials of the degree 2k with respect to ϵ', ϵ'', and e'. These polynomials contain the factor $\epsilon'^{|j'|}\epsilon''^{|j''|}$ provided $j' + j''$ is even and the factor $e'\epsilon'^{|j'|}\epsilon''^{|j''|}$ if $j' + j''$ is odd. The other factor is a polynomial in ϵ'^2, ϵ''^2, and e'^2.

The mean absolute motion n can be expanded in the form of

$$n = n^{(0)} + n^{(2)} + n^{(4)} + n^{(6)} + \cdots. \tag{20}$$

The quantity $n^{(2k)}$ ($k = 2, 3, \ldots$) which has the expression

$$n^{(2k)} = -\mu^k\,[[g^{(k-2)}]], \qquad (k = 2, 3, \ldots),$$

is of the order of magnitude 2k. This is a polynomial of the degree $k - 1$ /78 with respect to ϵ'^2, ϵ''^2, and e'^2; the quantities $n^{(2)}$, $n^{(4)}$, ... all contain μ as factor. We have, specifically,

$$\begin{aligned}
n^{(0)} &= x_1^{*-3}, \qquad n^{(2)} = -\mu\,\frac{d\,\tilde{\mathfrak{F}}_{0,0}^{1,0,0,0}}{dx_1^*},\\
n^{(4)} &= -\mu^2\,[[g^{(0)}]] = -\mu^2\,(G_{0,0}^{0,0,0} + G_{0,0}^{0,2,0}\,\varrho_*'^2 + G_{0,0}^{0,0,2}\,\varrho_*''^2)\\
&= -\mu^2\,\frac{d\,\tilde{\mathfrak{F}}_{0,0}^{2,0,0,0}}{dx_1^*} - \mu\,\varepsilon'^2\,\frac{d\,\tilde{\mathfrak{F}}_{0,0}^{1,0,2,0}}{dx_1^*} - \mu\,\varepsilon''^2\,\frac{d\,\tilde{\mathfrak{F}}_{0,0}^{1,0,0,2}}{dx_1^*}\\
&\quad - \mu\,e'^2\left\{\frac{d\,\tilde{\mathfrak{F}}_{0,0}^{1,2,0,0}}{dx_1^*} - 2\,\frac{d\,\tilde{\mathfrak{F}}_{1,0}^{1,1,1,0}}{dx_1^*}\,\frac{\tilde{\mathfrak{F}}_{1,0}^{1,1,1,0}}{\tilde{\mathfrak{F}}_{0,0}^{1,0,2,0}}\right.\\
&\qquad\qquad\left. + \frac{d\,\tilde{\mathfrak{F}}_{0,0}^{1,0,2,0}}{dx_1^*}\left(\frac{\tilde{\mathfrak{F}}_{1,0}^{1,1,1,0}}{\tilde{\mathfrak{F}}_{0,0}^{1,0,2,0}}\right)^2\right\}.
\end{aligned} \tag{21}$$

In this Section, we have assumed that the divisors (1) are not small. Let us see what happens if one of these divisors, for example, $r\nu_0' + (r + s)\nu_0''$ becomes small. For abbreviation, we will put

$$r\,\nu'_0 + (r + s)\,\nu''_0 = \sigma. \tag{22}$$

In view of the method used in forming the coefficients of the expansions (2) and (13) it is easy to show that the functions φ_k', ψ_k', φ_k'', ψ_k'', and $g^{(k)}$ contain the factor

$$\sigma^{-k},$$

and that the quantities ν_k' and ν_k'' contain

$$\sigma^{-k+1}$$

and, finally, that the coefficient $\delta^{(k)}$ of the expansion (17) contains

$$\sigma^{-k-1}.$$

It results from this also that certain of the coefficients $A^{(2k+1)}_{j', j''}$ and $B^{(2k+1)}_{j', j''}$ /79
in the series (12) are comparable in magnitude to

$$\mu^{k+\frac{1}{2}}\sigma^{-k}$$

and that we have coefficients $C^{(2k)}_{j', j''}$ of the series (19) that are comparable to

$$\mu^{k}\sigma^{-k}.$$

Thus, it is quite obvious that the expansions of this Section are still valid if all the divisors (1) are of the order of $\sqrt{\mu}$ or larger but that these expansions are illusory as soon as any of these divisors become of the order of μ or smaller.

We will state that a characteristic planet of the type $\frac{p+1}{p}$ is "regular" if the divisors (1) are sufficiently large so that the first terms of the series (12) and (19) converge sufficiently rapidly but that such a planet is "singular" in the opposite case.

For singular planets, the quantity Δ, defined by eq.(6) of Section 19 (by putting there q = 1), approximately satisfies an equation

$$r\,\mathfrak{F}^{1,0,2,0}_{0,0} + (r+s)\,\mathfrak{F}^{1,0,0,2}_{0,0} = 0, \tag{23}$$

where r and s are two whole numbers. In the expressions for the coefficients $\mathfrak{F}^{1,0,2,0}_{0,0}$ and $\mathfrak{F}^{1,0,0,2}_{0,0}$ [see eq.(3) of Section 19 and the equations given on pp.125and 126),we must introduce

$$x_1 = \left(\frac{p+1}{p} + \sqrt{\mu}\,\Delta\right)^{-\frac{1}{3}} = \left(\frac{p}{p+1}\right)^{\frac{1}{3}}\left\{1 - \frac{1}{3}\frac{p}{p+1}\sqrt{\mu}\,\Delta + \cdots\right\}.$$

Let $F^{1,0,2,0}_{0,0,0,0}$, $F^{1,0,0,2}_{0,0,0,0}$, etc.,be the values of the coefficients $F^{1,0,2,0}_{0,0,0,0}$, $F^{1,0,0,2}_{0,0,0,0}$ etc., calculated with the value $x_1 = \left(\frac{p}{p+1}\right)^{\frac{1}{3}}$. By neglecting μ in eq.(23), we /80
can consider the equation

$$-s\,F^{1,0,2,0}_{0,0,0,0} + r\,\frac{3}{\Delta^2}\left(\frac{p+1}{p}\right)^{\frac{4}{3}}\left(F^{1,0,1,0}_{-p,p+1,1,0}\right)^2 + \sqrt{\mu}\,\frac{\Phi(\Delta^2)}{\Delta^5} = 0,$$

where $\varphi(\Delta^2)$ is a certain polynomial of the third degree in Δ^2 with numerical coefficients. Still neglecting μ, we will obtain

$$\Delta = \pm \Delta_{p,r,s} + \Delta'_{p,r,s}\sqrt{\mu} + \cdots,$$

where $\Delta_{p,r,s}$ and $\Delta'_{p,r,s}$ are certain numerical constants that depend only on the three whole numbers p, r, and s.

For singular planets we must have

$$\Delta = \pm \Delta_{p,r,s} + \Delta'_{p,r,s}\sqrt{\mu} + C\mu, \tag{24}$$

where C is an arbitrary constant whose absolute value is not too great.

Evidently, the quantity $\Delta^2_{p,r,s}$ has the following expression

$$\Delta^2_{p,r,s} = \frac{3r}{s}\left(\frac{p+1}{p}\right)^{\frac{4}{3}} \frac{(\bar{F}^{1,0,1,0}_{-p,p+1,1,0})^2}{\bar{F}^{1,0,2,0}_{0,0,0,0}}, \tag{25}$$

or else, with the Gylden coefficients $\gamma_k^{i,j}$,

$$\Delta^2_{p,r,s} = \frac{6r}{s}\left(\frac{p+1}{p}\right)^2 \frac{[3(p+1)\gamma_0^{1,p+1} + 2\gamma_1^{1,p+1}]^2}{3\gamma_1^{1,0} + 4\gamma_2^{1,0}}. \tag{25'}$$

In the computation of $\gamma_k^{i,j}$ we must use the following value for the ratio of the major axes:

$$\alpha = \left(\frac{p}{p+1}\right)^{\frac{2}{3}}.$$

The expression for $\Delta'_{p,r,s}$ obviously is more complicated.

Section 23.

In Sections 19 - 22, we gave the theory of the secular variations of characteristic regular planets, by assuming that the eccentricities and the inclination are comparable in magnitude to $\sqrt{\mu}$ or smaller. We encountered there singular planets in the cases in which $q \geq 3$ and also in the case in which $q = 1$. /81
For $q = 2$, all planets whose eccentricity and inclination are of the order of $\sqrt{\mu}$ (or smaller) are regular.

In this Section 23, we will investigate the planets for which ρ_1^* or ρ_2^* (or both) are comparable in magnitude to $\mu^{\frac{1}{4}}$. For reasons of symmetry, it is convenient to consider, at least formally, not only ρ_1^* and ρ_2^* but also the eccentricities e' as quantities of the order of $\mu^{\frac{1}{4}}$.

We will start from eqs.(33) and (38) of Section 17 which, after introduction of the independent variable

$$t_1 = \mu t \qquad (1)$$

can be written as

$$\frac{d\xi_k^*}{dt_1} = \frac{d}{d\eta_k^*}\,\frac{F^* - F_0^*}{\mu}, \quad \frac{d\eta_k^*}{dt_1} = -\frac{d}{d\xi_k^*}\,\frac{F^* - F_0^*}{\mu}, \quad (k=1,2) \qquad (2)$$

$$\frac{d(y_1^* - x_1^{*-3}t)}{dt_1} = -\frac{d}{dx_1^*}\,\frac{F^* - F_0^*}{\mu}. \qquad (3)$$

Evidently, the equations of secular variations, in the theory of ordinary planets, can be given a similar form [see eqs.(1) and (2) of Section 3].

A considerable analogy exists between the expansions of the function

$$\frac{F^* - F_0^*}{\mu},$$

in the theory of ordinary planets and in the theory of characteristic planets. In fact, in the first expansion, the quantities

$$\mu, \qquad F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2}$$

correspond to the quantities /82

$$\sqrt{\mu}, \qquad f_{j_1,j_2}^{i,\bar{m},m_1,m_2}$$

in the second expansion.

In addition, the integers i, $\bar{m}$, m_1, m_2, j_1, j_2 in the two expansions satisfy the same conditions (11) of Section 3 and (29) of Section 17.

Finally, to the fundamental relation

$$F_{0,0,0,0}^{1,0,2,0} + F_{0,0,0,0}^{1,0,0,2} \equiv 0$$

in the theory of ordinary planets there corresponds, in the theory of characteristic planets of the types

$$\frac{p+q}{p} \qquad (q \geq 2),$$

the identical relation

$$f_{0,0}^{1,0,2,0} + f_{0,0}^{1,0,0,2} \equiv 0.$$

Conversely, for characteristic planets of the type

$$\frac{p+1}{p},$$

we have

$$f_{0,0}^{1,0,2,0} + f_{0,0}^{1,0,0,2} \neq 0.$$

According to the above statements, complete correspondence must exist between the theory of secular variations of ordinary planets, for which the eccentricities and the inclination are comparable to

$$\sqrt{\mu}$$

and the theory of secular variations of characteristic planets (where $q \geq 2$), for which the eccentricities and the inclination are of the order of

$$\mu^{\frac{1}{4}}.$$

In Sections 3 - 6 of Part I, we expressed the following unknowns for the ordinary planets: /83

$$\xi_k^*, \qquad \eta_k^* \qquad\qquad (k=1, 2)$$

and

$$y_1^* - x_1^{*-3} t$$

as functions of t_1 and of six integration constants

$$x_1^*, \quad \varepsilon', \quad \varepsilon'', \quad \gamma', \quad \gamma'' \text{ and } c.$$

We now return to eqs.(3) and (26) of Section 6. The expansions given in that Section still depend in a known manner on the eccentricity e' of the orbit of Jupiter and also on the quantities

$$\mu, \quad F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2}, \quad \frac{d}{dx^*} F_{0,0,j_1,j_2}^{i,\bar{m},m_1,m_2}. \tag{4}$$

The quantities ε', ε'', and e' are comparable in magnitude to $\sqrt{\mu}$.

For obtaining the corresponding solution of eqs.(2) and (3), i.e., the secular variations of characteristic planets (assuming $q \geq 2$), it is obviously sufficient to replace everywhere, in the expansions (3) and (26) of Section 6, the quantities (4) by

$$\sqrt{\mu}, \quad f_{j_1,j_2}^{i,\bar{m},m_1,m_2}, \quad \frac{d}{dx_1^*} f_{j_1,j_2}^{i,\bar{m},m_1,m_2} \tag{5}$$

and to consider ε', ε'', and e' as being of the order of $\mu^{\frac{1}{4}}$. All that we have stated above on the coefficients $\bar{\xi}$, $A_{j',j''}^{(2k+1)}$, $B_{j',j''}^{(2k+1)}$, $C_{j',j''}^{(2k)}$ of eqs.(3) and (26) of Section 6 (cf. pp.36-38 and 44-45 of Part I) remains valid also for the corresponding coefficients in the theory of the investigated characteristic planets if we everywhere write $\sqrt{\mu}$ instead of μ.

In the expansions of the unknowns ξ_1^*, η_1^*, and γ_1^* we have introduced a /84 divisor which is denoted by δ. For "ordinary" planets, the expression of δ can be written as follows [see eq.(2) of Section 6 and eqs.(1), (2), (3), of Section 7]:

$$\delta = -2\{2F_{0,0,0,0}^{1,0,4,0} + F_{0,0,0,0}^{1,0,2,2}\}\varepsilon'^2 - 2\{2F_{0,0,0,0}^{1,0,0,4} + F_{0,0,0,0}^{1,0,2,2}\}\varepsilon''^2$$

$$-2\left\{F_{0,0,0,0}^{1,2,2,0} + F_{0,0,0,0}^{1,2,0,2} - 4F_{0,0,1,0}^{1,1,3,0}\frac{F_{0,0,1,0}^{1,1,1,0}}{F_{0,0,0,0}^{1,0,2,0}} + 4F_{0,0,0,0}^{1,0,4,0}\left(\frac{F_{0,0,1,0}^{1,1,1,0}}{F_{0,0,0,0}^{1,0,2,0}}\right)^2\right\}e'^2$$

$$-2\{F_{0,0,0,0}^{2,0,2,0} + F_{0,0,0,0}^{2,0,0,2}\}\mu.$$

So as to make the expansions (3) and (26) of Section 6 valid, we had to assume that δ is not too small but comparable in magnitude to μ.

For characteristic planets (assuming that $q \geq 2$), the corresponding divisor will have the expression

$$\delta = -2\{2f_{0,0}^{1,0,4,0} + f_{0,0}^{1,0,2,2}\}\varepsilon'^2 - 2\{2f_{0,0}^{1,0,0,4} + f_{0,0}^{1,0,2,2}\}\varepsilon''^2$$

$$-2\left\{f_{0,0}^{1,2,2,0} + f_{0,0}^{1,2,0,2} - 4f_{1,0}^{1,1,3,0}\frac{f_{1,0}^{1,1,1,0}}{f_{0,0}^{1,0,2,0}} + 4f_{0,0}^{1,0,4,0}\left(\frac{f_{1,0}^{1,1,1,0}}{f_{0,0}^{1,0,2,0}}\right)^2\right\}e'^2 \tag{6}$$

$$-2\{f_{0,0}^{2,0,2,0} + f_{0,0}^{2,0,0,2}\}\sqrt{\mu}.$$

To have the resultant expansions be valid, it must be assumed that this quantity is not too small but comparable to $\sqrt{\mu}$.

In Section 18, we have expressed the coefficients $f_{j_1,j_2}^{i,\bar{s},s_1,s_2}$ by means of the coefficients $F_{j_1,j_2}^{i,\bar{s},s_1,s_2}$ of Section 16. By making use of these expressions, we can distinguish two cases depending on whether $q \geq 3$ or $q = 2$.

Let us first assume that

$$q \geq 3.$$

Then the expression (6) for δ becomes /85

$$\delta = -2\{2F_{0,0,0,0}^{1,0,4,0} + F_{0,0,0,0}^{1,0,2,2}\}\varepsilon'^2 - 2\{2F_{0,0,0,0}^{1,0,0,4} + F_{0,0,0,0}^{1,0,2,2}\}\varepsilon''^2$$

$$-2\left\{F_{0,0,0,0}^{1,2,2,0} + F_{0,0,0,0}^{1,2,0,2} - 4F_{0,0,1,0}^{1,1,3,0}\frac{F_{0,0,1,0}^{1,1,1,0}}{F_{0,0,0,0}^{1,0,2,0}} + 4F_{0,0,0,0}^{1,0,4,0}\left(\frac{F_{0,0,1,0}^{1,1,1,0}}{F_{0,0,0,0}^{1,0,2,0}}\right)^2\right\}e'^2.$$

The coefficients of ϵ'^2 and ϵ''^2 are still negative, according to what we had demonstrated in Section 7. Consequently, the divisor δ truly will be of the order of $\sqrt{\mu}$ as long as ϵ' or ϵ'' (or both) are comparable in magnitude to $\mu^{\frac{1}{4}}$. Actually, the term in e'^2 is relatively small since e', in reality, is comparable in magnitude to $\sqrt{\mu}$. In addition, it can be demonstrated that the coefficient of e'^2 is also always negative. Thus, the characteristic planets, for which we have $q \geq 3$, are of the "ordinary" type if the eccentricity or the inclination (or both) are comparable in magnitude to $\mu^{\frac{1}{4}}$.

Let us then assume that

$$q = 2.$$

The expression (6) then becomes (neglecting the term in e'^2 which, in reality, is relatively small)

$$\begin{aligned} \delta = &-\left\{4 F^{1,0,4,0}_{0,0,0,0} + 2 F^{1,0,2,2}_{0,0,0,0} + \frac{12}{\Delta^2 x^{*4}} (F^{1,0,2,0}_{-p,p+2,2,0})^2\right\} \varepsilon'^2 \\ &-\left\{4 F^{1,0,0,4}_{0,0,0,0} + 2 F^{1,0,2,2}_{0,0,0,0} + \frac{12}{\Delta^2 x_1^{*4}} (F^{1,0,0,2}_{-p,p+2,0,2})^2\right\} \varepsilon''^2 \\ &+ \frac{16}{p\Delta} \{(F^{1,0,2,0}_{-p,p+2,2,0})^2 + (F^{1,0,0,2}_{-p,p+2,0,2})^2\} \sqrt{\mu} + \cdots. \end{aligned} \tag{7}$$

The coefficients of this expression are comparable in magnitude to unity. The coefficients of ϵ'^2 and ϵ''^2 are always negative; the coefficient of $\sqrt{\mu}$ has the same sign as the quantity Δ defined by eq.(6) of Section 19.

We state that a characteristic planet of the type $\frac{p+2}{p}$ is "regular" if the quantity δ of eq.(7) is of the order of $\sqrt{\mu}$; we state that the planet is "singular" if this divisor δ is of the order of $\mu^{3/4}$ or smaller. /86

In view of this, we can conclude that the characteristic planets of the type $\frac{p+2}{p}$ are regular if Δ is negative; for the singular planets, the quantity Δ is necessarily positive so that ϵ' or ϵ'' must be of the order of $\mu^{\frac{1}{4}}$. These singular planets have a mean absolute motion greater than $\frac{p+2}{p}$.

We would like to make a brief remark on the secular variations of characteristic planets of the type

$$\frac{p+1}{p}$$

while considering the eccentricities and the inclination as quantities of the order of $\mu^{\frac{1}{4}}$.

We will no longer start from the expansion

$$\frac{1}{\mu}(F^* - F_0^*) = \sum \mathfrak{F}_{j_1, j_2}^{i, \overline{m}, m_1, m_2} \mu^{i-1} e'^{\overline{m}} \varrho_1^{*m_1} \varrho_2^{*m_2} \cos(j_1 \omega_1^* + j_2 \omega_2^*)$$

(as had been done in Section 19) but from the analogous expansion

$$\frac{1}{\mu}(F^* - F_0^*) = \sum f_{j_1, j_2}^{i, \overline{m}, m_1, m_2} \sqrt{\mu}^{i-1} e'^{\overline{m}} \varrho_1^{*m_1} \varrho_2^{*m_2} \cos(j_1 \omega_1^* + j_2 \omega_2^*)$$

[see eqs.(4) and (28) in Section 17].

In Sections 19 and 22, we expressed the unknowns

$$\xi_k^*, \qquad \eta_k^* \qquad\qquad (k = 1, 2), \tag{8}$$

and

$$y_1^* - x_1^{*-3} t \tag{9}$$

as functions of t_1 and of six integration constants

$$x_1^*,\ \varepsilon',\ \varepsilon'',\ \gamma',\ \gamma'' \text{ and } c$$

[see eqs.(12), (19), (20), and (21) of Section 22]. The series found there /87
include also, in a known manner, the eccentricity e' as well as the quantities

$$\mu, \qquad \mathfrak{F}_{j_1, j_2}^{i, \overline{m}, m_1, m_2}, \qquad \frac{d}{dx_1^*} \mathfrak{F}_{j_1, j_2}^{i, \overline{m}, m_1, m_2}. \tag{10}$$

The quantities ε', ε'', and e' are comparable in magnitude to $\sqrt{\mu}$.

To obtain the corresponding solution in the case in which the eccentricities and the inclination are comparable to $\mu^{\frac{1}{4}}$, it is sufficient to replace, in the mentioned expressions of the unknowns (8) and (9), the quantities (10) by the corresponding quantities

$$\sqrt{\mu}, \qquad f_{j_1, j_2}^{i, \overline{m}, m_1, m_2}, \qquad \frac{d}{dx_1^*} f_{j_1, j_2}^{i, \overline{m}, m_1, m_2} \tag{11}$$

and to consider, finally, the quantities ε', ε'', and e' as being of the order of $\mu^{\frac{1}{4}}$.

The ratio of the consecutive terms in the resultant series is of the order of $\mu^{\frac{1}{2}}$, provided that none of the divisors

$$j' \nu'_0 + j'' \nu''_0 \tag{12}$$

becomes too small.

Let us now suppose that one of these divisors, for example,

$$r\nu'_0 + (r+s)\nu''_0 = 2sF^{1,0,2,0}_{0,0,0,0} - \frac{6r}{\Delta^2 x_1^{*4}}(F^{1,0,1,0}_{-p,p+1,1,0})^2 = \sigma \tag{13}$$

is small. The expansions of the unknowns (8) and (9) proceed in reality according to whol powers of the ratic

$$\sqrt{\mu} : \sigma.$$

We can conclude from this that the considered expansions certainly become illusory if one of the divisors (12), for example σ, is of the order of $\sqrt{\mu}$ or smaller.

As in Section 22, we state here that a characteristic planet of the type $\frac{p+1}{p}$ is "regular" if the divisors (12) are sufficiently large so that the first terms of the mentioned expansions converge sufficiently rapidly but that such a planet is "singular" in the opposite case. /88

For singular planets, the quantity Λ defined by eq.(6) of Section 19, differs only by a quantity of the order of $\sqrt{\mu}$ having a value of

$$\pm \Delta_{p,r,s},$$

where the quantity $\Delta^2_{p,r,s}$ is also defined by eqs.(25) or (25') of Section 22.

In Sections 20 - 23, we integrated the equations of the secular variations for characteristic and regular minor planets. It is now easy to find, for these planets, the definite form of the canonical variables x_1, y_1, ξ_k, η_k defined in Section 2.

In fact, we have first shown that the differences

$$x_1 - \dot{x}_1,\quad y_1 - \dot{y}_1,\quad \xi_k - \dot{\xi}_k,\quad \eta_k - \dot{\eta}_k$$

can be expanded in the form of eq.(16) of Section 16. Then, we have shown that the differences

$$\dot{x}_1 - x_1^*,\quad \dot{y}_1 - y_1^*,\quad \dot{\xi}_k - \xi_k^*,\quad \dot{\eta}_k - \eta_k^*$$

can be expanded in the form of eq.(40) of Section 17. Here, the quantity x_1^* is an arbitrary constant. Finally, we have expressed

$$y_1^* - (nt + c),\quad \xi_k^*,\quad \eta_k^*$$

as trigonometric functions of the two linear arguments of w' and w''. It follows from this that the unknowns

$$x_1, \quad y_1 - (nt + c), \quad \xi_k, \quad \eta_k$$ /89

can be expanded in trigonometric series in multiples of the four linear arguments with respect to time:

$$nt + c, \quad t, \quad w' = \mu \nu' t + \gamma', \quad w'' = \mu \nu'' t + \gamma''.$$

We do not believe it necessary to give here such a detailed discussion of the theory of singular characteristic minor planets. It is necessary to state only that this problem can be reduced to one degree of freedom and that the principles of Part II of this research are applicable, with certain modifications.

RESEARCH ON THE MOTION OF MINOR PLANETS **/1

Part IV

H.v.Zeipel*

The term "critical of the type $(p + q):p$" will be used here to denote minor planets for which the mean motion differs from the rational number $(p + q):p$ by a quantity which is comparable in magnitude to the mass μ of Jupiter or smaller.

From the viewpoint of practical calculation of the perturbations, the theory of critical planets offers relatively little interest since such planetoids, actually, occur quite rarely in nature. It is well known from statistics that the large gaps in the distribution of the asteroids are located exactly at the spots where the period of revolution with respect to that of Jupiter would approach the numbers $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{7}{3}$. In this vicinity, it seems that no planets at all are present. The most remarkable critical planets are (279) Thule which is the only one in the vicinity of the commensurability $\frac{4}{3}$ and (153) Hilda with several companions near the commensurability $\frac{3}{2}$. However, the largest number /2 of critical planets correspond to the ratios $\frac{7}{2}, \frac{8}{3}, \frac{9}{4}, \frac{11}{5}, \ldots$, in which the numerator differs from the denominator by a number which is larger than 4.

How can these gaps be explained? Can a planet, placed into these lacunae, maintain itself there? Can the critical planets be transformed into comets because of the presence of perturbations? Only a rigorous theory could furnish a complete answer to these interesting questions. However, for a rigorous theory, the presently available analytical methods seem insufficient.

For the time being, one must be satisfied with a formal theory.

In what follows, we will make an attempt to base a formal theory of the critical planets on trigonometric and semiconvergent series, such as we have used previously for the theory of ordinary or characteristic planets. In this Part IV, we will limit the discussion to critical planets of the types $(p + q):p$, by assuming that

$$q \geq 3.$$

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Among these planets, the following two types must be differentiated: "singly critical" planets, for which the slowly varying portion of the major axis is not close to certain values dependent on p and q and on two other integers α and β; "doubly critical" planets, for which the secular portion of the major axis approaches such a value.

In the case of singly critical planets, only a single small divisor exists. The problem is thus reduced to a canonical system with only one degree of freedom. Therefore, a formal integration is entirely possible. We have found that the major axis, the eccentricity, and the inclination remain more or less invariant. Thus, the motion of singly critical planets is stable from the viewpoint of formal calculus. It is possible that a libration exists between the longitude of the node and the longitude of the eccentric vector, as is the case for ordinary planets and for characteristic planets. However, the mean longitudes never take part in a libration.

The problem of doubly critical planets is characterized by two small divisors. Consequently, this problem is reduced to two degrees of freedom; a formal integration is not possible in complete generality. However, an integration can be performed under certain conditions which, themselves, are rather general. In the resultant solution, the major axis, the eccentricity, and the inclination remain more or less invariant such that formal stability is guaranteed. Occasionally, there is a libration between the longitude of the node and the longitude of the eccentric vector. A libration may also exist between these longitudes and the two mean longitudes or only between the two mean longitudes. Finally, it may happen that two librations take place simultaneously. /13

To investigate the formal stability of the doubly critical planets, it is not necessary that a formal integration be possible. By means of the first Jacobi integral, we will demonstrate that the motion of doubly critical planets is stable from the viewpoint of formal calculus, provided that

$$q \geq 5.$$

It seems that we can thus explain why there are no gaps in the asteroid belt, for $q \geq 5$.

Frequently, the motion is stable also for $q = 3$ or 4. However, for these values of q it has never been possible to establish stability for all values of the integration parameters. On the other hand, it has never been possible to demonstrate the existence of instability. Nevertheless, it is of interest that the ring of minor planets always shows gaps wherever the number q is equal to 3 or 4.

Section 24.

To study the motion of critical minor planets, it is convenient to start from the canonical system (22) of Section 16, which is written as follows:

$$\frac{d\dot{x}_1}{dt}=\frac{d\dot{F}}{d\dot{y}},\quad \frac{d\dot{y}}{dt}=-\frac{d\dot{F}}{d\dot{x}_1},$$

$$\frac{d\dot{\xi}_k}{dt}=\frac{d\dot{F}}{d\dot{\eta}_k},\quad \frac{d\dot{\eta}_k}{dt}=-\frac{d\dot{F}}{d\dot{\xi}_k}. \qquad (k=1,\,2). \tag{1}$$

The characteristic function can be expanded in the form of /4

$$\dot{F}=\dot{F}_0+\mu\dot{F}_1+\mu^2\dot{F}_2+\cdots.$$

Here, we have specifically

$$\dot{F}_0=\frac{1}{2\dot{x}_1^2}+\frac{p+q}{p}\dot{x}_1. \tag{2}$$

In addition, by putting

$$\dot{\xi}_k=\dot{\varrho}_k\cos\dot{\omega}_k,\quad \dot{\eta}_k=\dot{\varrho}_k\sin\dot{\omega}_k, \qquad (k=1,\,2),$$

the expansions of $\dot{F}_i$ are written as

$$\dot{F}_i=\sum F^{i,\bar{m},m_1,m_2}_{\iota p,-\iota(p+q),j_1,j_2}\,e^{\bar{m}}\,\dot{\varrho}_1^{m_1}\,\dot{\varrho}_2^{m_2}\cos(\iota p\dot{y}+j_1\dot{\omega}_1+j_2\dot{\omega}_2). \tag{3}$$

The coefficients $F^{i,\bar{m},m_1,m_2}_{\iota p,-\iota(p+q),j_1,j_2}$ are certain functions of $\dot{x}_1$ defined already in Section 16. In the sum (3), the exponents $\bar{m}$, m_1, m_2 and the indices ι, j_1, j_2 all take integral values that satisfy the conditions

$$|j_1|\leq\leq m_1,\quad |j_2|\leq\leq m_2=\text{even},$$

$$|\iota q+j_1+j_2|\leq\leq\bar{m}+2i-2, \tag*{*(4)}$$

from which it follows that

$$|\iota q|\leq\leq\bar{m}+m_1+m_2+2i-2. \tag{5}$$

In the theory of critical minor planets, a differentiation must be made between the various types characterized by the value of the whole number q.

This Part IV of our research will be devoted to the types in which

$$q\geq 3.$$

---

* By the notation $a\leq\leq b$, we mean that $b-a$ is an even nonnegative integer.

In accordance with eqs.(3), (4), and (5) we will then have, up to terms of the second degree incl., with respect to the quantities $\dot\rho_1$, $\dot\rho_2$, e', and $\sqrt{\mu}$, /5

$$
\begin{aligned}
\dot F_1 &= F^{1,0,0,0}_{0,0,0,0} + F^{1,0,2,0}_{0,0,0,0}\dot\varrho_1^2 + F^{1,0,0,2}_{0,0,0,0}\dot\varrho_2^2 \\
&\quad + 2F^{1,1,1,0}_{0,0,1,0}e'\dot\varrho_1\cos\dot\omega_1 + F^{1,2,0,0}_{0,0,0,0}e'^2 + \cdots, \\
\mu\dot F_2 &= \mu F^{2,0,0,0}_{0,0,0,0} + \cdots.
\end{aligned}
$$

In the types in which $q \geq 3$, it is easy to reduce the system (1) to the normal form discussed in Section 1. It is sufficient to introduce new variables

$$
z,\ \xi',\ \xi'';\ \dot y,\ \eta',\ \eta''
$$

by putting

$$
\begin{aligned}
\dot x_1 &= \bar x_1 + \mu z, & \dot y &= \dot y, \\
\dot\xi_1 &= \bar c e' + \sqrt{\mu}\,\xi', & \dot\eta_1 &= \sqrt{\mu}\,\eta', \\
\dot\xi_2 &= \sqrt{\mu}\,\xi'', & \dot\eta_2 &= \sqrt{\mu}\,\eta''.
\end{aligned}
\tag{6}
$$

Here, the numerical constants $\bar x_1$ and $\bar c$ have the following expressions:

$$
\begin{aligned}
\bar x_1 &= \left(\frac{p}{p+q}\right)^{1/3}, \\
\bar c &= -F^{1,1,1,0}_{0,0,1,0}(\bar x_1) : F^{1,0,2,0}_{0,0,0,0}(\bar x_1).
\end{aligned}
\tag{7}
$$

We will here consider the eccentricity e' of the orbit of Jupiter as being of the order of $\sqrt{\mu}$ and thus set

$$
e' = \sqrt{\mu}\,e_0. \tag{8}
$$

The characteristic function F, expressed by the new variables and expanded in powers of $\sqrt{\mu}$, can then be given the following form:

$$
F = \frac{1}{2\bar x_1^2} + \frac{p+q}{p}\bar x_1 + \mu F^{1,0,0,0}_{0,0,0,0}(\bar x_1) + \mu^2 H. \tag{9}
$$

The function H, defined in this manner, is given by an expansion /6

$$
H = H^{(0)} + \mu^{1/2}H^{(1/2)} + \mu H^{(1)} + \mu^{3/2}H^{(3/2)} + \cdots. \tag{10}
$$

By putting, in addition,

$$t_1 = \mu t,$$

the new variables will satisfy the canonical system

$$\begin{aligned}
\frac{dz}{dt_1} &= \frac{dH}{dy}, & \frac{dy}{dt_1} &= -\frac{dH}{dz},\\
\frac{d\xi'}{dt_1} &= \frac{dH}{d\eta'}, & \frac{d\eta'}{dt_1} &= -\frac{dH}{d\xi'},\\
\frac{d\xi''}{dt_1} &= \frac{dH}{d\eta''}, & \frac{d\eta''}{dt_1} &= -\frac{dH}{d\xi''}.
\end{aligned} \tag{11}$$

It is easy to write the expression for $H^{(0)}$. Making use of the notations

$$h(z) = \frac{3}{2x_1^4} z^2 + \frac{dF^{1,0,0,0}_{0,0,0,0}(\bar{x}_1)}{dx_1} z + \text{const.}, \tag{12}$$
$$\bar{\nu}' = -2F^{1,0,2,0}_{0,0,0,0}(x_1), \quad \bar{\nu}'' = -2F^{1,0,0,2}_{0,0,0,0}(\bar{x}_1),$$

we actually obtain

$$H^{(0)} = h(z) - \frac{\bar{\nu}'}{2}(\xi'^2 + \eta'^2) - \frac{\bar{\nu}''}{2}(\xi''^2 + \eta''^2). \tag{13}$$

Thus, the system (11) actually has the normal form already treated in Section 1.

The numerical quantities $\bar{\nu}'$ and $\bar{\nu}''$ satisfy the relation

$$\bar{\nu}' + \bar{\nu}'' = 0.$$

We will give the general expression of the functions $H^{(n)}$ that appear in the expansion (10). For this purpose, we will set 17

$$\begin{aligned}
\xi' &= \varrho' \cos\omega', & \eta' &= \varrho' \sin\omega',\\
\xi'' &= \varrho'' \cos\omega'', & \eta'' &= \varrho'' \sin\omega'',
\end{aligned} \tag{14}$$

and consider the sum

$$\begin{aligned}
f^{(m)} = \sum \frac{d^s}{dx_1^s} & F^{i,\bar{m},m_1,m''}_{\iota p,-\iota(p+q),j_1,j''}(\bar{x}_1)\\
& \cdot \frac{z^s}{s!} \varrho_0^{\bar{m}} \varrho'^{m_1} \varrho''^{m''} \cos(\iota p y + j_1\omega' + j''\omega''),
\end{aligned} \tag{15}$$

where the indices i, s, $\bar{m}$, m_1, m'', ι, j_1, j'' run through all integral values that satisfy the conditions

$$i \geq 1, \quad s \geq 0, \quad \overline{m} \geq 0, \quad m_1 \geq 0, \quad m'' \geq 0,$$
$$i + s + \frac{1}{2}(\overline{m} + m_1 + m'') - 2 = m,$$
$$|j_1| \leq\leq m_1, \quad |j''| \leq\leq m'' = \text{even},$$
$$|\iota q + j_1 + j''| \leq\leq \overline{m} + 2i - 2. \tag{16}$$

By virtue of eqs.(14), the quantity $f^{(s)}$ will be a certain function $f^{(s)}(\xi', \xi'', \eta', \eta'')$ of the variables $\xi', \xi'', \eta', \eta''$.

Since the substitution (6) as well as eq.(9) are given, it is obvious that we will have, in the expansion (10),

$$H^{(k)} = \frac{(-1)^k}{2} \frac{k+3}{\bar{x}_1^{k+4}} z^{k+2} + f^{(k)}(\bar{c}e_0 + \xi', \xi'', \eta', \eta''),$$
$$(k = 0, 1, 2, \ldots) \tag{17}$$
$$H^{(k+\frac{1}{2})} = f^{(k+\frac{1}{2})}(\bar{c}e_0 + \xi', \xi'', \eta', \eta'').$$

If we wish to substitute $\overline{ce_0} + \xi'$ for ξ' in the function $f^{(s)}(\xi', \xi'', \eta', \eta'')$, it is first necessary to form the successive derivatives of the function

$$\varphi_{j_1}^{(m_1)} = \varrho'^{m_1} \cos(j_1 \omega' + a)$$

with respect to ξ'. (Here, we denoted by a an arbitrary quantity independent of ρ' and ω'.)

First, we have the differential formulas /8

$$\frac{d}{d\varrho'} = \cos\omega' \frac{d}{d\xi'} + \sin\omega' \frac{d}{d\eta'},$$
$$\frac{d}{\varrho' d\omega'} = -\sin\omega' \frac{d}{d\xi'} + \cos\omega' \frac{d}{d\eta'},$$

from which the inverse formula

$$D = \frac{d}{d\xi'} = \cos\omega' \frac{d}{d\varrho'} - \sin\omega' \frac{d}{\varrho' d\omega'}.$$

is obtained.

By applying successively this formula and by putting, for abbreviation,

$$\alpha_1 = \frac{m_1 + j_1}{2}, \quad \beta_1 = \frac{m_1 - j_1}{2},$$

it is easily found that

$$
\begin{aligned}
D\varphi_{j_1}^{(m_1)} &= \alpha_1\varphi_{j_1-1}^{(m_1-1)} + \beta_1\varphi_{j_1+1}^{(m_1-1)},\\
D^2\varphi_{j_1}^{(m_1)} &= \alpha_1(\alpha_1-1)\varphi_{j_1-2}^{(m_1-2)} + 2\alpha_1\beta_1\varphi_{j_1}^{(m_1-2)} + \beta_1(\beta_1-1)\varphi_{j_1+2}^{(m_1-2)},\\
D^3\varphi_{j_1}^{(m_1)} &= \alpha_1(\alpha_1-1)(\alpha_1-2)\varphi_{j_1-3}^{(m_1-3)} + 3\alpha_1(\alpha_1-1)\beta_1\varphi_{j_1-1}^{(m_1-3)}\\
&\quad + 3\alpha_1\beta_1(\beta_1-1)\varphi_{j_1+1}^{(m_1-3)} + \beta_1(\beta_1-1)(\beta_1-2)\varphi_{j_1+3}^{(m_1-3)}.
\end{aligned}
$$

The general law here is manifest. We can set

$$\frac{d^{m_1-m'}\varrho'^{m_1}\cos(j_1\omega'+a)}{(m_1-m')!\,d\xi'^{m_1-m'}} = \sum_{j'} c_{j_1,j'}^{m_1,m'}\varrho'^{m'}\cos(j'\omega'+a). \tag{18}$$

In this sum, the integer j' takes all values for which

$$j'-j_1+m_1-m' = 0,\ 2,\ 4,\ \dots\ 2(m_1-m'). \tag{19}$$

It is easy to demonstrate that

$$c_{j_1,j'}^{m_1,m'} = \binom{\alpha_1}{\alpha'}\binom{\beta_1}{\beta'} = \frac{\alpha_1!}{(\alpha_1-\alpha')!\,\alpha'!}\,\frac{\beta_1!}{(\beta_1-\beta')!\,\beta'!} \tag{20}$$

with the notations

$$\alpha' = \frac{m'+j'}{2}, \qquad \beta' = \frac{m'-j'}{2}. \tag{21}$$

Now, we can write eqs.(17) in the form of

$$
\begin{aligned}
H^{(k)} &= \frac{(-1)^k}{2}\frac{k+3}{x_1^{k+4}}z^{k+2}\\
&\qquad + \sum H_{\iota,j',j''}^{k,m',m''}\varrho'^{m'}\varrho''^{m''}\cos(\iota p\dot{y}+j'\omega'+j''\omega''),
\end{aligned}
\tag{22}
$$

$$H^{(k+\frac{1}{2})} = \sum H_{\iota,j',j''}^{k+\frac{1}{2},m',m''}\varrho'^{m'}\varrho''^{m''}\cos(\iota p\dot{y}+j'\omega'+j''\omega'').$$

In view of this, eqs.(15), (17), (18), and (20) show that

$$H_{\iota,j',j''}^{m,m',m''} = \sum_{i,s,\bar{m},\alpha_1,\beta_1}\frac{d^s}{dx_1^s}F_{\iota p,-\iota(p+q),j_1,j''}^{i,\bar{m},m_1,m''}(\bar{x}_1)\cdot\binom{\alpha_1}{\alpha'}\binom{\beta_1}{\beta'}\bar{c}^{m_1-m'}e_0^{\bar{m}+m_1-m'}\frac{z^s}{s!}. \tag{23}$$

Here, we used the notations

$$m_1 = \alpha_1 + \beta_1, \quad j_1 = \alpha_1 - \beta_1,$$
$$m' = \alpha' + \beta', \quad j' = \alpha' - \beta',$$
$$m'' = \alpha'' + \beta'', \quad j'' = \alpha'' - \beta''.$$

Finally, in accordance with the relations (16) and (19), the indices i, s, $\bar{m}$, α_1, β_1 take all values that satisfy the conditions

$$\begin{gathered} i \geq 1, \quad s \geq 0, \quad \bar{m} \geq 0, \quad \alpha_1 > \alpha', \quad \beta_1 > \beta' \\ |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta''| \leq\leq \bar{m} + 2i - 2 \\ = 2m + 2 - (\alpha_1 + \beta_1 + \alpha'' + \beta'') - 2s. \end{gathered} \tag{24}$$

We will demonstrate that the indices m', m", ι, j', j" that appear in the polynomials (23) satisfy the conditions

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$$\begin{gathered} |j'| \leq\leq m', \quad |j''| \leq\leq m'' = \text{even}, \\ |\iota q + j' + j''| + m' + m'' \leq\leq 2m + 2 - 2s \leq\leq 2m + 2, \\ |\iota q| \leq\leq 2m + 2 - 2s \leq\leq 2m + 2. \end{gathered} \tag{25}$$

The two first of these relations are evident. The others are true since, on the one hand,

$$\begin{aligned} &|\iota q + \alpha' - \beta' + \alpha'' - \beta''| \\ &= |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta'' - (\alpha_1 - \alpha') + (\beta_1 - \beta')| \\ &\leq\leq |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta''| + \alpha_1 - \alpha' + \beta_1 - \beta' \\ &\leq\leq 2m + 2 - (\alpha' + \beta' + \alpha'' + \beta'') - 2s, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} |\iota q| &= |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta'' - (\alpha_1 - \beta_1) - (\alpha'' - \beta'')| \\ &\leq\leq |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta''| + |\alpha_1 - \beta_1| + |\alpha'' - \beta''| \\ &\leq\leq |\iota q + \alpha_1 - \beta_1 + \alpha'' - \beta''| + \alpha_1 + \beta_1 + \alpha'' + \beta'' \\ &\leq\leq 2m + 2 - 2s. \end{aligned}$$

Below, we will give all values of the indices m', m", j', j" that satisfy the conditions

$$|j'| \leq\leq m', \quad |j''| \leq\leq m'' = \text{even},$$

as well as the relation appearing at the head of each of the following Tables.

Table 1: $|j' + j''| + m' + m'' = 0$.

m'	m''	j'	j''
0	0	0	0

Table 2: $|j' + j''| + m' + m'' = 2$.

m'	m''	j'	j''
2	0	0	0
1	0	± 1	0
0	2	0	0

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Table 3: $|j' + j''| + m' + m'' = 4$.

m'	m''	j'	j''
4	0	0	0
3	0	± 1	0
2	0	± 2	0
2	2	0	0
2	2	± 2	∓ 2
1	2	± 1	0
1	2	± 1	∓ 2
0	2	0	± 2
0	4	0	0

Table 4: $|j' + j''| + m' + m'' = 6$.

m'	m''	j'	j''
6	0	0	0
5	0	± 1	0
4	0	± 2	0
3	0	± 3	0
4	2	0	0
4	2	± 2	∓ 2
3	2	± 3	∓ 2
3	2	± 1	∓ 2
3	2	± 1	0
2	2	± 2	0
2	2	0	± 2
1	2	± 1	± 2
2	4	0	0
2	4	± 2	∓ 2
1	4	± 1	0
1	4	± 1	∓ 2
0	4	0	± 2
0	6	0	0

Table 5: $|3 + j' + j''| + m' + m'' = 3$. /12

m'	m''	j'	j''
3	0	−3	0
2	0	−2	0
1	0	−1	0
0	0	0	0
1	2	−1	−2
0	2	0	−2

Table 6: $|3 + j' + j''| + m' + m'' = 5$.

m'	m''	j'	j''
5	0	−3	0
4	0	−4	0
4	0	−2	0
3	0	−1	0
2	0	0	0
1	0	1	0.
3	2	−3	0
3	2	−1	−2
2	2	−2	0
2	2	0	−2
2	2	−2	−2
1	2	−1	0
1	2	+1	−2
0	2	0	0
1	4	−1	−2
1	4	+1	−4
0	4	0	−4
0	4	0	−2

Table 7: $|4 + j' + j''| + m' + m'' = 4$.

m'	m''	j'	j''
4	0	−4	0
3	0	−3	0
2	0	−2	0
1	0	−1	0
0	0	0	0
2	2	−2	−2
1	2	−1	−2
0	2	0	−2
0	4	0	−4

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Table 8: $|4 + j' + j''| + m' + m'' = 6.$

m'	m''	j'	j''
6	0	−4	0
5	0	−5	0
5	0	−3	0
4	0	−2	0
3	0	−1	0
2	0	0	0
1	0	+1	0
4	2	−4	0
4	2	−2	−2
3	2	−3	−2
3	2	−3	0
3	2	−1	−2
2	2	−2	0
2	2	0	−2
1	2	−1	0
1	2	+1	−2
0	2	0	0
2	4	−2	−2
2	4	0	−4
1	4	−1	−4
1	4	−1	−2
1	4	+1	−4
0	4	0	−2
0	6	0	−4

Table 9: $|5 + j' + j''| + m' + m'' = 5.$ /14

m'	m''	j'	j''
5	0	−5	0
4	0	−4	0
3	0	−3	0
2	0	−2	0
1	0	−1	0
0	0	0	0
3	2	−3	−2
2	2	−2	−2
1	2	−1	−2
0	2	0	−2
1	4	−1	−4
0	4	0	−4

Table 10: $|6 + j' + j''| + m' + m'' = 6.$

m'	m''	j'	j''
6	0	-6	0
5	0	-5	0
4	0	-4	0
3	0	-3	0
2	0	-2	0
1	0	-1	0
0	0	0	0
4	2	-4	-2
3	2	-3	-2
2	2	-2	-2
1	2	-1	-2
0	2	0	-2
2	4	-2	-4
1	4	-1	-4
0	4	0	-4
0	6	0	-6

Section 25. /15

To further reduce the canonical system (11) of Section 24, we will apply the method given in Section 1.

The integration divisors, resulting from this method, obviously are given by the formula

$$\iota p \nu(z) + (j' - j'') \bar{\nu}',$$

where we have put

$$\nu(z) = -\frac{dh(z)}{dz}. \tag{1}$$

If $\iota = 0$ and $j' = j''$, the corresponding divisors will cancel out.

Let, now, α be an arbitrary positive whole number and let β be an arbitrary integer having no factor in common with α. Let us denote by

$$\bar{z}_{\alpha,\beta}$$

the root of the equation linear in z

$$\alpha p \nu(z) + \beta \bar{\nu}' = 0.$$

Since the expressions of $h(z)$ and $\bar{\nu}'$ are given [see eqs.(12) of Section 24], the numerical value of $\bar{z}_{\alpha,\beta}$ can be calculated by means of the formula

$$\bar{z}_{\alpha,\beta} = -\frac{\bar{x}_1^4}{3}\left\{\frac{d}{dx_1} F^{1,0,0,0}_{0,0,0,0}(\bar{x}_1) + \frac{2\beta}{p\alpha} F^{1,0,2,0}_{0,0,0,0}(x_1)\right\}. \tag{2}$$

It is necessary to differentiate between singly critical planets for which the initial value of z is not close to a value $\bar{z}_{\alpha\beta}$ and doubly critical planets of the case (α, β) for which the initial value of $z - \bar{z}_{\alpha\beta}$ is sufficiently small.

In this Section, we will roughly sketch the theory of singly critical planets. In this case, we have only a single small divisor, namely, the one that corresponds to $i = 0$ and $j' = j''$. Making use of the method given in Section 1, the considered canonical system can be reduced to another system with one degree of freedom.

To effect this reduction, we must start from the characteristic /16

$$H(z, \xi', \xi''; y, \eta', \eta'')$$

as well as from the unknown function

$$H_*(z, \xi', \xi''; \eta', \eta'')$$

and then use these for forming the equation of partial derivatives

$$H\left(\frac{dS}{dy}, \frac{dS}{d\eta'}, \frac{dS}{d\eta''}; y, \eta', \eta''\right) = H_*\left(z, \xi', \xi''; \frac{dS}{d\xi'}, \frac{dS}{d\xi''}\right). \tag{3}$$

The functions H, H_*, and S are expanded in the form of

$$\begin{aligned} H &= H^{(0)} + \mu^{1/2}H^{(1/2)} + \mu H^{(1)} + \mu^{3/2}H^{(3/2)} + \mu^2 H^{(2)} + \cdots, \\ H_* &= H_*^{(0)} + \mu H_*^{(1)} + \mu^2 H_*^{(2)} + \cdots, \\ S &= S^{(0)} + \mu^{1/2}S^{(1/2)} + \mu S^{(1)} + \mu^{3/2}S^{(3/2)} + \mu^2 S^{(2)} + \cdots. \end{aligned} \tag{4}$$

These series must be introduced into eq.(3), after which the expansions of the two members in powers of $\mu^{1/2}$ must be compared.

By putting

$$\begin{aligned} H_*^{(0)} &= H^{(0)} = h(z) - \frac{\bar{\nu}'}{2}(\xi'^2 + \eta'^2) - \frac{\bar{\nu}''}{2}(\xi''^2 + \eta''^2), \\ S^{(0)} &= z\dot{y} + \xi'\eta' + \xi''\eta'', \quad S = S^{(0)} + S_1, \end{aligned} \tag{5}$$

eq.(3) will be satisfied for $\mu = 0$.

After this, eq.(3) can be writtin in the following form:

$$H\left(z+\frac{dS_1}{dy},\ \xi'+\frac{dS_1}{d\eta'_i},\ \xi''+\frac{dS_1}{d\eta''_i};\ y,\ \eta',\ \eta''_i\right)$$
$$=H_*\left(z,\ \xi',\ \xi'';\ \eta'_i+\frac{dS_1}{d\xi'},\ \eta''_i+\frac{dS_1}{d\xi''}\right). \tag{6}$$

By here comparing the coefficients of $\mu^{1/2}$ in the expansions of the two members, we obtain the equation /17

$$-\nu(z)\frac{dS^{(1/2)}}{dy}-\sum{}''\bar{\nu}'\xi'\frac{dS^{(1/2)}}{d\eta'_i}+H^{(1/2)}=-\sum{}''\bar{\nu}'\eta'_i\frac{dS^{(1/2)}}{d\xi'}.$$

(The sign Σ'' in front of a given term designates a sum of the two terms, of which the second is obtained from the term written by priming the letters twice instead of only once.) It is convenient to introduce the variables ρ', ρ'', ω', and ω'' defined by eqs.(14) of Section 24. To abbreviate the notations, we will also make use of the symbolic notation

$$Df=\nu(z)\frac{df}{dy}+\bar{\nu}'\frac{df}{d\omega'}+\bar{\nu}''\frac{df}{d\omega''}. \tag{7}$$

In view of this, the equation which yields $S^{(1/2)}$ is written simply

$$DS^{(1/2)}=H^{(1/2)}. \tag{8}$$

In the function $H^{(1/2)}$, the index m has the value $\frac{1}{2}$. The last of the relations in the system (25) of Section 24 then demonstrates that

$$|q|\leq 3.$$

Consequently, we will have

$$H^{(1/2)}\equiv 0, \qquad \text{if } q\geq 4,$$

and, in accordance with the second formula of the system (22) of Section 24,

$$H^{(1/2)}=2\sum H_{1,j',j''}^{1/2,m',m''}\rho'^{m'}\rho''^{m''}\cos(py+j'\omega'+j''\omega'') \qquad \text{if } q=3. \tag{9}$$

For q = 3, the above sum includes six terms that correspond to the six combinations given in Table 5 of Section 24.

It should finally be mentioned that the function $H^{(1/2)}$ does not depend on the variable z. This results from the last of the relations in the system (25) of Section 24, which actually shows that the exponent s assumes exclusively the value zero in the expression of the coefficients $H_{1,j',j''}^{1/2,m',m''}$ given by eq.(23) of Section 24.

Now, eq.(8) yields /18

$$S^{(1/2)} \equiv 0, \qquad \text{if } q \geq 4, \tag{10}$$

and

$$S^{(1/2)} = 2 \sum \frac{H_{1,j',j''}^{1/2,m',m''} \varrho'^{m'} \varrho''^{m''}}{p\nu(z) + (j' - j'')\nu'} \sin(p\dot{y} + j'\omega' + j''\omega''), \qquad \text{if } q = 3. \tag{11}$$

A comparison of the coefficients of μ in the two members of eq.(6) will yield the equation

$$DS^{(1)} = \tilde{H}^{(1)} - H_*^{(1)}, \tag{12}$$

where

$$\tilde{H}^{(1)} = H^{(1)} + \sum'' \frac{dH^{(1/2)}}{d\xi'} \frac{dS^{(1/2)}}{d\eta'} + \frac{3}{2x_1^4} \left(\frac{dS^{(1/2)}}{d\dot{y}}\right)^2$$

$$+ \sum'' \frac{\bar{\nu}'}{2} \left\{ \left(\frac{dS^{(1/2)}}{d\xi'}\right)^2 - \left(\frac{dS^{(1/2)}}{d\eta'}\right)^2 \right\}. \tag{13}$$

Now, let f be an arbitrary function periodic with respect to $p\dot{y}$, ω', and ω'' with the period 2π and expandable in a trigonometric series in arguments of the form $\iota p\dot{y} + j'\omega' + j''\omega''$. In this Section, we will denote by

$$[f]$$

the ensemble of the terms of f where

$$\iota = 0, \qquad j' = j''.$$

In view of this, it is sufficient to put, in eq.(12),

$$H_*^{(1)} = [\tilde{H}^{(1)}]$$

so as to finally obtain the function $S^{(1)}$ without small divisors.

Evidently, we can continue in this manner and successively determine the various terms of the expansions (4).

It is easy to demonstrate that, in all the arguments /19

$$\iota p\dot{y} + j'\omega' + j''\omega''$$

which appear in the trigonometric and finite expansions of the functions

$$\tilde{H}^{(m)}, \; H_*^{(m)} \text{ and } S^{(m)},$$

the number $2q$ is always of the same parity as $2m$, exactly as had been the case for the various arguments of the function $H^{(*)}$.

Because of this, we have always been able to put

$$H_*^{(1/2)} = H_*^{(3/2)} = H_*^{(5/2)} = \cdots = 0.$$

From this it also follows that

$$\left.\begin{aligned} H^{(1/2)} = H^{(3/2)} = H^{(5/2)} = \cdots = 0, \\ S^{(1/2)} = S^{(3/2)} = S^{(5/2)} = \cdots = 0, \end{aligned}\right\} \text{ if } q \text{ is even.}$$

We will examine the function $H_*^{(1)}$ in more detail.

If

$$q \geq 4,$$

we will have

$$S^{(1/2)} = 0, \quad \tilde{H}^{(1)} = H^{(1)}.$$

To obtain the expression for $H_*^{(1)}$, it is sufficient to retain, in $H^{(1)}$, all terms in the arguments of which we have $\iota = 0$, $j' = j''$. In accordance with Tables 1, 2, and 3 of Section 24, this will yield the following expression:

$$\begin{aligned} H_*^{(1)} = H_{0,0,0}^{1,4,0}\varrho'^4 + H_{0,0,0}^{1,2,2}\varrho'^2\varrho''^2 + H_{0,0,0}^{1,0,4}\varrho''^4 \\ + H_{0,0,0}^{1,2,0}\varrho'^2 + H_{0,0,0}^{1,0,2}\varrho''^2 + H_{0,0,0}^{1,0,0}. \end{aligned} \tag{14}$$

Then, the function $H_*^{(1)}$ is a polynomial of the second degree with respect to ρ'^2 and ρ''^2; this polynomial does not depend on the angle $\omega' + \omega''$.

Let us now see what will happen if /20

$$q = 3.$$

In all arguments of the various terms of the functions (9) and (11) and of their derivatives, we will then have

$$-3 \leq j' + j'' \leq 0.$$

This is demonstrated by the six combinations in Table 5 of Section 24. Consequently, in the arguments independent of $\dot{y}$ in the trigonometric expansion of the function $H^{(1)} - \tilde{H}^{(1)}$ defined by eq.(13), the indices j' and j'' satisfy the condition

$$|j' + j''| \leq 3.$$

Thus, in the arguments of the function $H_*^{(1)} - [H^{(1)}]$ where $\iota = 0$ and $j' = j'' =$ = even, we will have

$$|2j''| \leq 3,$$

whence

$$j' = j'' = 0.$$

In addition, we have seen above that the function $[H^{(1)}]$ is independent of ω' and ω''. Consequently, if $q = 3$, the function $H_*^{(1)}$ will also be a polynomial in ρ'^2 and ρ''^2. However, this polynomial is now obviously of the third degree.

We will also investigate the function $H_*^{(2)}$.

So as not to complicate the discussion excessively, we will investigate in detail only the types in which

$$q \geq 4.$$

Then, the equation which yields $H_*^{(2)}$ and $S^{(2)}$ can be written as follows:

$$DS^{(2)} = \tilde{H}^{(2)} - H_*^{(2)},$$

where we have made use of the notation /21

$$\tilde{H}^{(2)} = H^{(2)} + \frac{dH^{(1)}}{dz}\frac{dS^{(1)}}{d\dot{y}} + \sum'' \frac{dH^{(1)}}{d\xi'}\frac{dS^{(1)}}{d\eta'} + \frac{3}{2x_1^4}\left(\frac{dS^{(1)}}{d\dot{y}}\right)^2$$
$$+ \sum'' \frac{\nu'}{2}\left\{\left(\frac{dS^{(1)}}{d\xi'}\right)^2 - \left(\frac{dS^{(1)}}{d\eta'}\right)^2\right\} - \sum'' \frac{dH_*^{(1)}}{d\eta'}\frac{dS^{(1)}}{d\xi'}. \qquad (15)$$

The sought expression for $H_*^{(2)}$ will then be

$$H_*^{(2)} = [\tilde{H}^{(2)}].$$

The terms of the function $[H^{(2)}]$ correspond to those of the combinations given in Tables 1 - 4 of Section 24, where $j' = j''$. In all these combinations, we have $j' = j'' = 0$. Thus, the function $[H^{(2)}]$ is a polynomial in ρ'^2 and ρ''^2, independent of $\omega' + \omega''$. The polynomial in question, obviously, is of the third degree with respect to ρ'^2 and ρ''^2.

In the terms of the derivative $\frac{dH^{(1)}}{dz}$, we will have, as in the part of $H^{(1)}$ that depends on z,

$$|\iota q| \leq \leq 2,$$

from which it follows that $\iota = 0$. This derivative thus is independent of $\dot{y}$. On the other hand, all arguments of the function $\frac{dS^{(1)}}{d\dot{y}}$ depend on $\dot{y}$. Consequently, we will have

$$\left[\frac{dH^{(1)}}{dz}\frac{dS^{(1)}}{dy}\right]\equiv 0.$$

To go further, we assume first that

$$q\geq 5.$$

We state that the functions $H^{(1)}$ and $S^{(1)}$ are then independent of $\dot{y}$ since we have there $|\iota q| \ll 4$ from which it follows that $\iota = 0$. All terms of the functions $H^{(1)}$, $H_*^{(1)}$, and $S^{(1)}$ then correspond to the combinations of the indices m', m'', j', j'' entered in Tables 1 - 3 of Section 24. In the derivatives of these functions with respect to ξ', η', ξ'', η'', we will have

$$|j'+j''|\leq 2.$$

The quantity $|j' + j''| = 2$ can occur only in the derivatives of terms that correspond to the combinations given on lines 2, 6, and 7 of Table 3. By retaining only the terms of the derivatives of $H^{(1)}$ and of $S^{(1)}$ where $|j' + j''| = 2$, it is easy to find that the terms of the function $H_*^{(2)}$, where we would have $|j' + j''| = 4$, will cancel out. /22

Thus, above $q \geq 5$, the function $H_*^{(2)}$ is a polynomial in ρ'^2 and ρ''^2 which, incidentally, is of the third degree.

We will assume that

$$q=4,$$

then $H^{(1)}$ and $S^{(1)}$ will also contain terms that depend on $\dot{y}$. To obtain, in $[\tilde{H}^{(2)}]$, terms that depend on the argument $2\omega' + 2\omega''$, it is sufficient - in accordance with the above statements - to retain, in $H^{(1)}$ and $S^{(1)}$, only the terms in which

$$\iota=1,\ \ |4+j'+j''|+m'+m''=4.$$

These terms correspond to the combinations given in Table 7 of Section 24. In the corresponding terms of the derivatives of $H^{(1)}$ and $S^{(1)}$, with respect to any one of the variables ξ', η', ξ'', η'', we will have

$$\iota=1,\ \ -3\leq j'+j''\leq 0.$$

In the squares and the two-to-two products of these derivatives, we will thus have $|j' + j''| \leq 3$ for each argument in which $\iota = 0$ and, consequently, $j' = j'' = 0$ in any argument in which $\iota = 0$ and $j' = j''$ = even.

Thus, we see that the terms of the function $H_*^{(2)}$, which depend on the argument $2\omega' + 2\omega''$, all are contained in the function

$$\frac{3}{2x_1^4}\left[\left(\frac{dS^{(1)}}{d\dot{y}}\right)^2\right].$$

To obtain these terms, it is sufficient - in accordance with Table 7 of Section 24 - to retain, in the derivative of $S^{(1)}$, with respect to $\dot{y}$ only the terms

$$\frac{2}{\nu(z)} H^{1,2,2}_{1,-2,-2} \varrho'^2 \varrho''^2 \cos(p\dot{y} - 2\omega' - 2\omega'') + \frac{2}{\nu(z)} H^{1,0,0}_{1,0,0} \cos p\dot{y}.$$

Thus, for the types in which $q = 4$, we will have /23

$$H_*^{(2)} = \frac{6}{x_1^4(\nu(z))^2} H^{1,2,2}_{1,-2,-2} H^{1,0,0}_{1,0,0} \varrho'^2 \varrho''^2 \cos(2\omega' + 2\omega'') + P(\varrho'^2, \varrho''^2),$$

where $P(\rho'^2, \rho''^2)$ is a polynomial of the fourth degree in ρ'^2 and ρ''^2 which, incidentally, is independent of ω' and ω''.

In studying the function $H_*^{(1)}$, we are limited to types in which $q \geq 4$. If $q = 3$, the discussion would become much more complicated. Nevertheless, by canceling the coefficients

$$H^{1,3,0}_{1,-3,0},\ H^{1,1,2}_{1,-1,-2} \text{ and } H^{1,1,0}_{1,-1,0}$$

in the expression (9) for the function $H^{(1/2)}$, we found that the function $H_*^{(2)}$ actually contains a term with the argument $2\omega' + 2\omega''$.

Let us now assume that the functions

$$S(z, \xi', \xi''; \dot{y}, \eta', \eta'') \text{ and } H_*(z, \xi', \xi''; \eta', \eta'')$$

are known. Starting from the function

$$S(z_*, \xi'_*, \xi''_*; \dot{y}, \eta', \eta'') \tag{16}$$

we can form the canonical transformation

$$\begin{aligned} z &= \frac{dS}{d\dot{y}}, & \dot{y}_* &= \frac{dS}{dz_*}, \\ \xi' &= \frac{dS}{d\eta'}, & \eta'_* &= \frac{dS}{d\xi'_*}, \\ \xi'' &= \frac{dS}{d\eta''}, & \eta''_* &= \frac{dS}{d\xi''_*}, \end{aligned} \tag{17}$$

which, because of eqs.(5), can be also written as

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$$z - z_* = \frac{d(S - S^{(0)})}{d\dot{y}}, \quad \dot{y}_* - \dot{y} = \frac{d(S - S^{(0)})}{dz_*},$$

$$\xi' - \xi'_* = \frac{d(S - S^{(0)})}{d\eta'}, \quad \eta'_* - \eta' = \frac{d(S - S^{(0)})}{d\xi'_*}, \tag{18}$$

$$\xi'' - \xi''_* = \frac{d(S - S^{(0)})}{d\eta''}, \quad \eta''_* - \eta'' = \frac{d(S - S^{(0)})}{d\xi''_*}.$$

It is easy to solve these equations and to express the variables z, ξ', ξ''; $\dot{y}$, η', η'' as functions of the variables z_*, ξ'_*, ξ''_*; $\dot{y}_*$, η'_*, η''_*. Since the function (16) satisfies the equation of partial derivatives obtained from eq.(3) by writing there simply z_*, ξ''_*, ξ''_* instead of z, ξ', ξ'', it is obvious that we will obtain

$$H(z, \xi', \xi''; \dot{y}, \eta', \eta'') = H_*(z_*, \xi'_*, \xi''_*; \eta'_*, \eta''_*),$$

provided that, in the first term, the quantities z, ξ', ξ''; $\dot{y}$, η', η'' are replaced by their expressions as functions of the variables z_*, ξ'_*, ξ''_*; $\dot{y}_*$, η'_*, η''_*.

Consequently, in view of the canonical transformation (17), the canonical system (11) of Section 24 will be replaced by the equations

$$\frac{dz_*}{dt_1} = 0, \quad \frac{d\dot{y}_*}{dt_1} = -\frac{dH_*}{dz_*}, \tag{19}$$

$$\frac{d\xi'_*}{dt_1} = \frac{dH_*}{d\eta'_*}, \quad \frac{d\eta'_*}{dt_1} = -\frac{dH_*}{d\xi'_*},$$

$$\frac{d\xi''_*}{dt_1} = \frac{dH_*}{d\eta''_*}, \quad \frac{d\eta''_*}{dt_1} = -\frac{dH_*}{d\xi''_*}. \tag{20}$$

The quantity z_* is reduced to a constant, since the function H_* does not depend on the argument $\dot{y}_*$.

However, it is easy to obtain a further simplification and to reduce the canonical system formed by the four equations (20) to another canonical system which has only one degree of freedom. For this purpose, we set

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$$\xi'_* = \varrho'_* \cos \omega'_*, \quad \eta'_* = \varrho'_* \sin \omega'_*,$$

$$\xi''_* = \varrho''_* \cos \omega''_*, \quad \eta''_* = \varrho''_* \sin \omega''_*. \tag{21}$$

The characteristic function H_* of the system (20) depends on ω'_* and ω''_* only in the combination $\omega'_* + \omega''_*$. Consequently, instead of the variables

$$\xi'_*, \quad \eta'_*,$$
$$\xi''_*, \quad \eta''_*,$$

it is convenient to introduce first

$$\tfrac{1}{2}\rho'^2_*, \quad \omega'_*,$$
$$\tfrac{1}{2}\rho''^2_*, \quad \omega''_*,$$

and next

$$\tfrac{1}{2}\rho^2_* = \tfrac{1}{2}\rho'^2_*, \qquad \omega_* = \omega'_* + \omega''_*,$$
$$\tfrac{1}{2}\varkappa_* = \tfrac{1}{2}\rho'^2_* - \tfrac{1}{2}\rho''^2_*, \quad -\omega''_*.$$

The two transformations are canonical. Thus, we ultimately arrive at the following final equations:

$$z_* = \text{const.}, \quad \frac{dy_*}{dt_1} = -\frac{dH_*}{dz_*}, \tag{22}$$

$$\varkappa_* = \text{const.}, \quad \frac{d\omega'_*}{dt_1} = \frac{dH_*}{d\frac{1}{2}\varkappa_*}, \tag{23}$$

$$\frac{d\frac{1}{2}\rho^2_*}{dt_1} = \frac{dH_*}{d\omega_*}, \quad \frac{d\omega_*}{dt_1} = -\frac{dH_*}{d\frac{1}{2}\rho^2_*}. \tag{24}$$

After integration of the canonical system (24), which has only one degree of freedom, we can obtain the arguments $\dot{y}_*$ and ω''_* as functions of t_1 by quadratures, over the intermediary of eqs.(22) and (23).

Let us recall that the characteristic function H_* is given by the expansion

$$H_* = H^{(0)}_* + \mu H^{(1)}_* + \mu^2 H^{(2)}_* + \cdots.$$

Above, we had already discussed the character of the three first terms. /26

By making use of the definite variables, we obviously will have

$$H^{(0)}_* = h(z_*) - \tfrac{1}{2}\bar{\nu}'\varkappa_* = \text{const.}$$

The function $H^{(1)}_*$ is a polynomial in ρ^2_*, which is of the second degree if $q \geq 4$ and of the third degree if $q = 3$.

If $q \geq 5$, the function $H^{(2)}_*$ is independent of ω_* and, in addition, is a polynomial of the third degree in ρ^2_*. Conversely, for the types in which $q = 3$ or 4, the function $H^{(2)}_*$ is a linear function in $\cos 2\omega_*$ with coefficients that are polynomials in ρ^2_*.

In general, the functions $H_*^{(n)}$ are polynomials in ρ_*^2 and $\varkappa_*$; they are rational with respect to the constant z_* and are periodic with respect to the argument $2\omega_*$ with the period 2π.

Integration of the canonical system (24) is very easy since the derivative

$$\frac{dH_*^{(1)}}{d\frac{1}{2}\rho_*^2} \tag{25}$$

is not small for the initial value of $\frac{1}{2}\rho_*^2$. Then, we can again apply the method given in Section 1 and reduce the problem to zero degrees of freedom.

If, at the origin of time, the derivative (25) is small while the second derivative

$$\frac{d^2H_*^{(1)}}{(d\frac{1}{2}\rho_*^2)^2} \tag{26}$$

is comparable in magnitude to unity, we can proceed more or less in the same manner as that given in Sections 10 - 12 and integrate the system (24) by means of elliptic or trigonometric functions, depending on the various situations that might occur.

The derivatives (25) and (26) cannot become small simultaneously except in the case in which $q = 3$. An application of the Jacobi integration method, based on principles similar to those given in Section 10, will then lead to a /27 differential equation in S which is of the third degree with respect to the quantity $\frac{dS}{d\omega_*}$. In this case, the elliptic functions no longer are sufficient for integrating the system (24). Nevertheless, since this case is rather special, we do not believe it necessary to discuss it in further detail.

In addition, it is sufficient to make a few general remarks on the nature of the solution of eqs.(22), (23), and (24).

Let us primarily consider the first Jacobi integral

$$H_* = h.$$

Because of this integral, ρ_*^2 is a certain periodic function of ω_*, with the period π. Since $H_*^{(0)}$ is constant and since $H_*^{(1)}$ does not depend on ω_*, it is obvious that ρ_*^2 remains more or less constant. It is also evident that $\rho_*\cos\omega_*$ and $\rho_*\sin\omega_*$ are still certain finite and periodic functions with respect to time, having a period Π which is extremely long and at least comparable to μ^{-2}. The period Π is a certain function of the constant h (and of the parameters z_* and $\varkappa_*$), appearing in the Jacobi integral.

If the value of the derivative (25), which is more or less constant, is not too small, then the argument ω_* will possess a mean motion such that ω_* increases (or decreases) by 2π as soon as t increases by the period Π. For certain values of h, the period Π becomes infinitely large. In addition, if the value of h is located between certain limits, the mean velocity of the argument ω_* becomes zero; in this case, there will be libration and the argument ω_* will be a periodic function which oscillates between two extreme values.

Let us now return to the variables $\rho'_* = \rho_*$ and $\rho''_* = \sqrt{\rho^2_* - \varkappa_*}$. Evidently, these functions have the period Π and are more or less constant.

Let us now study the arguments $\dot{y}_*$, ω'_*, and ω''_*, considered as functions of time. After having expressed H_* as a function of the variables

$$z_*,\ \tfrac{1}{2}\varrho''^2_*,\ \tfrac{1}{2}\varrho'^2_*;\ \omega_*,$$

we will have /28

$$\frac{d\dot{y}_*}{dt} = -\mu\frac{dH_*}{dz_*},\quad \frac{d\omega'_*}{dt} = -\mu\frac{dH_*}{d\frac{1}{2}\varrho'^2_*},\quad \frac{d\omega''_*}{dt} = -\mu\frac{dH_*}{d\frac{1}{2}\varrho''^2_*}.$$

All these derivatives are periodic functions of t, with the period Π. Let us denote by

$$\mu\nu,\ \mu\nu',\ \mu\nu'' \tag{27}$$

the mean values of these periodic functions. In this case, the quantities (27) are the mean motions of the arguments

$$\dot{y}_*,\ \omega'_*,\ \omega''_*.$$

By neglecting μ, we will have

$$\begin{aligned} \nu &= \nu(z_*) + \cdots, \\ \nu' &= \bar{\nu}' + \cdots, \\ \nu'' &= \bar{\nu}'' + \cdots. \end{aligned} \tag{28}$$

We can then pass to the functions ξ'_*, ξ''_*; η'_*, η''_* which are given by eqs.(21). These functions are obviously finite and slowly variable.

Let us now return to the relations (18). These can be solved for the unknowns z, ξ', ξ''; $\dot{y}$, η', η'' in accordance with the Lagrange method, generalized to three variables. After performing this solution, we replace there z_*, ξ'_*, ξ''_*; $\dot{y}_*$, η'_*, η''_* by their expressions as functions of t. We thus find that the differences $z - z_*$, $\xi' - \xi'_*$, $\xi'' - \xi''_*$; $\dot{y} - \dot{y}_*$, $\eta' - \eta'_*$, $\eta'' - \eta''_*$ are small oscillating functions with slow variations. In the types in which q = 3, all these differences are of the order of $\mu^{1/2}$. If, conversely, $q \geq 4$, the differences $\xi' - \xi'_*$, $\xi'' - \xi''_*$; $\dot{y} - \dot{y}_*$, $\eta' - \eta'_*$, $\eta'' - \eta''_*$ are of the order of μ whereas $z - z_*$ is of the order of $\mu^{q/2-1}$. From this, we can conclude that the arguments $\dot{y}$, ω', and ω'' [where the two latter are defined by eqs.(14) of Section 24] have the quantities (27) as mean motions.

Finally, we must pass through the substitution (6) of Section 24, the /29
formula (36) of Section 17, and the transformation (13) of Section 16 to arrive ultimately at expressions for the primary variables x_1, ξ_1, ξ_2; y_1, η_1, η_2 defined in Section 2. We note here specifically that the differences $x_1 - \dot{x}_1$, $\xi_k - \dot{\xi}_k$; $y_1 - \dot{y}_1$, $\eta_k - \dot{\eta}_k$ are small oscillating functions being of the order of μ and possessing rapid variations.

The obtained results can be combined in the following formulas, which can serve also for classifying the various inequalities in the theory of singly critical planets:

$$
\begin{aligned}
x_1 &= \dot{x}_1 + (x_1 - \dot{x}_1) = \bar{x}_1 + \mu z + (x_1 - \dot{x}_1) \\
&= (\bar{x}_1 + \mu z_*) + \mu (z - z_*) + (x_1 - \dot{x}_1), \\
y_1 &= \dot{y}_1 + (y_1 - \dot{y}_1) = \dot{y} + \frac{p+q}{p} t + (y_1 - \dot{y}_1) \\
&= \left(\dot{y}_* + \frac{p+q}{p} t\right) + (\dot{y} - \dot{y}_*) + (y_1 - \dot{y}_1), \\
\xi_1 &= \dot{\xi}_1 + (\xi_1 - \dot{\xi}_1) = (\bar{c} e' + \sqrt{\mu}\, \xi'_*) + \sqrt{\mu}(\xi' - \xi'_*) + (\xi_1 - \dot{\xi}_1), \\
\eta_1 &= \dot{\eta}_1 + (\eta_1 - \dot{\eta}_1) = \sqrt{\mu}\, \eta'_* + \sqrt{\mu}(\eta' - \eta'_*) + (\eta_1 - \dot{\eta}_1), \\
\xi_2 &= \dot{\xi}_2 + (\xi_2 - \dot{\xi}_2) = \sqrt{\mu}\, \xi''_* + \sqrt{\mu}(\xi'' - \xi''_*) + (\xi_2 - \dot{\xi}_2), \\
\eta_2 &= \dot{\eta}_2 + (\eta_2 - \dot{\eta}_2) = \sqrt{\mu}\, \eta''_* + \sqrt{\mu}(\eta'' - \eta''_*) + (\eta_2 - \dot{\eta}_2).
\end{aligned}
\tag{27}
$$

The inequalities of each element are thus subdivided into three groups. The inequalities of each group are sufficiently well characterized by our above discussions.

In accordance with the definition of singly critical planets, the value of z and thus also the value of the parameter z_*, must avoid the vicinity of certain numerical values $\bar{z}_{\alpha\beta}$. We will express this condition in a different manner. Let

$$n, \quad n' = 1 \quad \text{and} \quad -\mu \nu''$$

be the mean motions of the mean longitude y_1 of the asteroid, of the mean longitude $y_2 = t$ of Jupiter, and of the longitude $\vartheta = -\omega_2$ of the ascending node /30
of the asteroid. For critical planets, the quantity

$$pn - (p+q) n'$$

is of the order of μ. For singly critical planets, there is no linear expression with integral coefficients α and β and of the form

$$\alpha \{pn - (p+q) n'\} - \beta \mu \nu'',$$

which would be small with respect to μ. (We assume that the values of α and β are not too large.)

Section 26.

In determining, in the preceding Section, the various terms of the function S, we encountered in each integration divisors of the form of

$$\iota p\nu(z) + (j' - j'')\bar{\nu}',$$

where the whole numbers ι and $j' - j''$ are not both zero. On terminating the series yielding S and H_* at a certain term, the introduced divisors will be finite in number and correspond to certain limited values of the numbers ι and $j' - j''$. We have assumed that these divisors, of finite number, are sufficiently large so that the first terms of the series converge sufficiently rapidly.

Let us now see under what conditions the series (4) of Section 25 no longer are applicable. This will happen as soon as any one of the divisors becomes too small. Let us assume that the value of μ is sufficiently small so that two divisors will not become too small simultaneously. In view of this, let

$$\nabla = \alpha p\nu(z) + \beta\bar{\nu}' = -\frac{3\alpha p}{x_1^4}(z - \bar{z}_{\alpha,\beta}) \tag{1}$$

be the unique small divisor. It is now necessary to find the greatest negative power of ∇ in the various terms of the series (4) of Section 25.

Let us first investigate the types in which /31

$$q = 3.$$

Primarily, we assume that the function $S^{(1/2)}$ is the first that is increased by the integration. In view of eq.(11) of the preceding Section for $S^{(1/2)}$, it is obvious that this will happen only if

$$\alpha = 1, \quad -3 \leq \beta \leq +2. \tag{2}$$

In the expression for $S^{(1/2)}$, the enlarged term includes ∇^{-1}. Equation (12) of Section 25, which gives $H_*^{(1)}$ and $S^{(1)}$, indicates that the principal terms of $H_*^{(1)}$ include ∇^{-2} and that the largest terms of $S^{(1)}$ contain ∇^{-3}. We have not written out the equation which determines the function $S^{(3/2)}$. However, it is obvious that the most expanded parts of the second member of this equation are products of a derivative of $S^{(1/2)}$ and a derivative of $S^{(1)}$. The highest negative power of ∇ in the expression for $S^{(3/2)}$ thus will be ∇^{-5}. The general law is obvious. Under the conditions in question here, the expansions (4) of Section 25 progress, in reality, in powers of the quantity

$$\frac{\mu^{1/2}}{(z - \bar{z}_{\alpha,\beta})^2}.$$

Secondly, let us assume that the small divisor appears for the first time in the function $S^{(1)}$. To define the conditions under which this might happen, it is necessary to study the arguments of the function $\tilde{H}^{(1)}$ given by eq.(13) of Section 25. In all arguments of the functions $H^{(1/2)}$ and $S^{(1/2)}$ as well as of their derivatives, we have, in accordance with Table 5 of Section 24,

$$\iota = 1, \quad -3 \leq j' - j'' \leq +2. \tag{3}$$

Consequently, in the arguments of the products of these pairwise derivatives, we will have either

$$\iota = 0, \quad -5 \leq j' - j'' \leq +5, \tag{4}$$

or /32

$$\iota = 2, \quad -6 \leq j' - j'' \leq +4. \tag{5}$$

In addition, Tables 1 - 3 of Section 24 show that the indices of the arguments of the function $H^{(1)}$ also satisfy the conditions (4). Thus, the arguments of the functions $\tilde{H}^{(1)}$ and $S^{(1)}$ are characterized by the conditions (4) and (5). It is therefore obvious that $S^{(1)}$ will be the first expanded function only if

$$\alpha = 2, \quad \beta = -5, -3, -1, +1, +3. \tag{6}$$

In the expression of $S^{(1)}$, the enlarged terms then include ∇^{-1}. Obviously, the function $H_*^{(1)}$ is not expanded in these cases.

To investigate the small divisors of the function $S^{(3/2)}$, it is not necessary to write the equation defining this function in great detail. It is sufficient to note that its second term is linear with respect to the derivatives of the expanded function $S^{(1)}$ and that, in the arguments of this second term, the number ι can assume only the values 1 or 3. From this, it follows that ∇^{-1} is encountered only in the expanded portion of the function $S^{(3/2)}$.

Let us now pass to the equation which furnishes $H_*^{(2)}$ and $S^{(2)}$. In the second member of this equation, the most expanded portions include the square of a derivative of $S^{(1)}$. Certain of these portions will be further expanded by the integration. This demonstrates that $S^{(2)}$ contains the small divisor raised to the third power and that $H_*^{(2)}$ contains its square.

It is easy to perceive the general law. In the cases (6), the series (4) of Section 25 are expanded, in reality, in powers of

$$\frac{\mu^{1/2}}{z - \bar{z}_{\alpha,\beta}}.$$

Thirdly, let us consider the cases in which $S^{(3/2)}$ is assumed to be the first expanded function. The conditions (3) are satisfied for the arguments of the functions $H^{(1/2)}$ and $S^{(1/2)}$ and of their derivatives; similarly, the conditions (4) or (5) are satisfied in the arguments of the functions $H^{(1)}$, $S^{(1)}$, $H_*^{(1)}$ and of their derivatives. In view of this and since also the form of the arguments of the function $H^{(3/2)}$ is given, it is obvious that, in the arguments of the function $S^{(3/2)}$ we will have either /133

$$\iota = 1, \quad -8 \leq j' - j'' \leq +7, \tag{7}$$

or

$$\iota = 3, \quad -9 \leq j' - j'' \leq +6. \tag{8}$$

Consequently, the function $S^{(3/2)}$ will be the first enlarged function only in the cases of

$$\alpha = 1, \ \beta = -8, -7, -6, -5, -4, +3, +4, +5, +6, +7, \tag{9}$$

$$\alpha = 3, \ \beta = -8, -7, -5, -4, -2, -1, +1, +2, +4, +5. \tag{10}$$

The principal terms of $S^{(3/2)}$ then contain ∇^{-1}.

The equation, yielding $H_*^{(2)}$ and $S^{(2)}$, has the form

$$DS^{(2)} = \tilde{H}^{(2)} - H_*^{(2)}.$$

In the expression for $\tilde{H}^{(2)}$, we will only write the expanded portion. For abbreviation, we make use of the symbolic notation

$$\Box f = \frac{3}{\bar{x}_1^4} \frac{dS^{(1/2)}}{d\dot{y}} \frac{df}{d\dot{y}} + \sum{}''' \bar{\nu}' \left\{ \frac{dS^{(1/2)}}{d\xi'} \frac{df}{d\xi'} - \frac{dS^{(1/2)}}{d\eta'} \frac{df}{d\eta'} \right\} + \sum{}'' \frac{dH^{(1/2)}}{d\xi'} \frac{df}{d\eta'}. \tag{11}$$

In addition, we will denote by

$$\breve{f}$$

the most expanded portion of a function f, i.e., the part of f which contains the highest negative power of ∇. In view of this, the expanded portion of $\tilde{H}^{(2)}$ will be

$$\Box \breve{S}^{(3/2)}. \tag{12}$$

In the arguments of the derivatives of $\breve{S}^{(3/2)}$, we have $\iota = \alpha$. In the arguments of the derivatives of $H^{(1/2)}$ and $S^{(1/2)}$, we always have $\iota = 1$. In the arguments of a given product in the expression (12), we will thus have $\iota = \alpha \pm 1$. So that eq.(12) can furnish the expanded terms of $H_*^{(2)}$ or the terms of $S^{(2)}$ doubly expanded by integration, it is necessary that $\alpha \pm 1$ be a multiple of α, i.e., that $\alpha = 1$. Thus, in the cases (10), the function $H_*^{(2)}$ is not at all enlarged /134 and the function $S^{(2)}$ includes only the power ∇^{-1}. We state that this is the same for the cases (9). In fact, in the arguments of $\frac{d\check{S}^{(3/2)}}{d\dot{y}}$ we then have

$$\iota = 1, \quad j' - j'' = \beta,$$

whereas we have

$$\iota = 1, \quad j' - j'' \neq \beta$$

in the arguments of $\frac{dS^{(1/2)}}{d\dot{y}}$. Similarly, in the arguments of $\check{S}^{(3/2)}$ with respect to ξ', ξ''; η' or η'', we have

$$\iota = 1, \quad j' - j'' = \beta \pm 1,$$

whereas we will have

$$\iota = 1, \quad j' - j'' \neq \beta \pm 1$$

in the arguments of the derivatives of $H^{(1/2)}$ and of $S^{(1/2)}$, with respect to these same variables. Consequently, in the arguments of the expression (12) it is never possible to have $\iota = s\alpha$ and $j' - j'' = s\beta$. Thus, also in the cases (9), the function $H_*^{(2)}$ is not enlarged at all and the function $\check{S}^{(2)}$ includes only ∇^{-1}.

The equation, yielding $S^{(5/2)}$, has the form

$$D\check{S}^{(5/2)} = \tilde{H}^{(5/2)}.$$

The second term is linear with respect to the derivatives of the expanded functions $S^{(3/2)}$ and $S^{(2)}$. It results from this that the function $S^{(5/2)}$ contains at most ∇^{-2}. It is easy to demonstrate that $S^{(5/2)}$ actually includes terms in ∇^{-2}. To prove this, it is sufficient to recall that the function $\tilde{H}^{(5/2)}$ contains the part /35

$$\sum{}''\left\{\frac{dH^{(1)}}{d\xi'}\frac{d\check{S}^{(3/2)}}{d\eta'} - \frac{dH_*^{(1)}}{d\eta'}\frac{d\check{S}^{(3/2)}}{d\xi'}\right\}.$$

In the functions $H^{(1)}$ and $H_*^{(1)}$ we will only show the terms

$$H^{1,2,0}_{0,0,0}\varrho'^2 + H^{1,0,2}_{0,0,0}\varrho''^2,$$

which are terms of the second degree, common to both $H^{(1)}$ and $H_*^{(1)}$. The resultant terms, in the function $\tilde{H}^{(5/2)}$, obviously are

$$2H^{1,2,0}_{0,0,0}\frac{d\check{S}^{(3/2)}}{d\omega'} + 2H^{1,0,2}_{0,0,0}\frac{d\check{S}^{(3/2)}}{d\omega''}.$$

The corresponding terms of $S^{(5/2)}$ are once more enlarged by integration and, consequently, include ∇^{-2}.

Let us now pass to the equation yielding $H_*^{(3)}$ and $S^{(3)}$:

$$DS^{(3)} = \bar{F}^{(3)} - H_*^{(3)}.$$

The most expanded portion of $\bar{H}^{(3)}$ obviously is

$$\Box \breve{S}^{(5/2)} + \frac{3}{2\bar{x}_1^4}\left(\frac{d\breve{S}^{(3/2)}}{dy}\right)^2 + \sum{}'' \frac{\bar{\nu}'}{2}\left\{\left(\frac{d\breve{S}^{(3/2)}}{d\xi'}\right)^2 - \left(\frac{d\breve{S}^{(3/2)}}{d\eta'}\right)^2\right\}. \quad (13)$$

We will demonstrate, as before, that the expression $\Box\breve{S}^{(5/2)}$ will no longer be expanded by the integration and that this same expression contributes nothing to the function $H_*^{(3)}$. The other parts of the expression (13), conversely, yield the terms $H_*^{(3)}$ which include ∇^{-2} and also the terms of $S^{(3)}$ enlarged three times by the integration.

It is easy to continue in this manner and to demonstrate that the principal parts of the functions $S^{(m)}$ and $H_*^{(m)}$ include the small divisor, as shown in the following Table:

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$S^{(3/2)}$, $S^{(2)}$	: ∇^{-1};	$H_*^{(2)}$:	∇^{0};
$S^{(5/2)}$	: ∇^{-2};		
$S^{(3)}$, $S^{(7/2)}$	: ∇^{-3};	$H_*^{(3)}$:	∇^{-2};
$S^{(4)}$	: ∇^{-4};	$H_*^{(4)}$:	∇^{-3};
$S^{(9/2)}$, $S^{(5)}$	: ∇^{-5};	$H_*^{(5)}$:	∇^{-4};
$S^{(11/2)}$	: ∇^{-6};		
$S^{(6)}$, $S^{(13/2)}$	: ∇^{-7};	$H_*^{(6)}$:	∇^{-6};
$S^{(7)}$	: ∇^{-8};	$H_*^{(7)}$:	∇^{-7};
$S^{(15/2)}$, $S^{(8)}$	: ∇^{-9};	$H_*^{(8)}$:	∇^{-8};

. . . .

Consequently, in the cases (9) and (10), the series (4) of Section 25 can be considered as actually expanded in powers of the quantity

$$\frac{\mu^{1/2}}{(z - \bar{z}_{\alpha,\beta})^2}.$$

Finally, we will treat the more general case in which the function $S^{(m)}$ $(m \geq 2)$ is assumed to be the first function expanded by the integration.

In this case, the principal part of $S^{(m)}$ includes ∇^{-1}.

The equation yielding $H_*^{(m+1/2)}$ and $S^{(m+1/2)}$ has the form

$$DS^{(m+\frac{1}{2})} = \tilde{H}^{(m+\frac{1}{2})} - H_*^{(m+\frac{1}{2})}.$$

The expanded portion of $\tilde{H}^{(m+1/2)}$ obviously is

$$\Box \check{S}^{(m)}.$$

As on p.200, we can demonstrate here that $H_*^{(m+1/2)}$ is not at all enlarged and that $S^{(m+1/2)}$ includes, at most, ∇^{-1}.

The equation, defining $H_*^{(m+1)}$ and $S^{(m+1)}$ is written as

$$DS^{(m+1)} = \tilde{H}^{(m+1)} - H_*^{(m+1)}.$$

Since $\tilde{H}^{(m+1)}$ is linear with respect to the derivatives of the expanded functions $S^{(m)}$ and $S^{(m+1/2)}$, it is obvious that $H_*^{(m+1)}$ can include at most ∇^{-1} and that $S^{(m+1)}$ can contain at most ∇^{-2}. As on p.201, we demonstrate that $S^{(m+1)}$ actually contains ∇^{-2}. /37

In continuing, it is easy to demonstrate, as we had done above for $H_*^{(m+1/2)}$ and $S^{(m+1/2)}$, that $H_*^{(m+3/2)}$ does not contain ∇^{-2} and that $S^{(m+3/2)}$ contains at most ∇^{-3}, and so on.

Thus, in all these latter cases, the series (4) of Section 25 are actually expanded in powers of the ratio

$$\frac{\mu}{z - \bar{z}_{\alpha,\beta}}.$$

Let us now investigate the influence of a small divisor ∇, for the case of types where

$$q = 4.$$

Then, the expansions yielding H and S include only whole powers of μ.

Let us first assume that a small divisor appears already in the function $S^{(1)}$. Table 7 of Section 24 indicates that this will happen only in the following cases:

$$\alpha = 1, \quad -4 \leq \beta \leq +4, \ \beta \neq +3. \tag{14}$$

In view of the general character of the functions $\tilde{H}^{(2)}$, $\tilde{H}^{(3)}$, ..., it is obvious that the series (4) of Section 25 progress then in powers of

$$\frac{\mu}{(z - \bar{z}_{\alpha,\beta})^2}.$$

Let us next assume that the small divisor is first encountered in the function $S^{(m)}$ $(m \geq 2)$.

The equation yielding $S^{(m+1)}$ has the form

$$DS^{(m+1)} = \tilde{H}^{(m+1)} - H_*^{(m+1)}.$$

It is necessary to fix the expanded portion of $\tilde{H}^{(m+1)}$. For abbreviation, it is convenient to introduce the notation /38

$$\Box' f = \frac{3}{\bar{x}_1^4}\frac{dS^{(1)}}{d\bar{y}}\frac{df}{d\bar{y}} + \sum{}'' \bar{\nu}'\left\{\frac{dS^{(1)}}{d\xi'}\frac{df}{d\xi'} - \frac{dS^{(1)}}{d\eta'}\frac{df}{d\eta'}\right\}$$
$$+ \frac{dH^{(1)}}{dz}\frac{df}{dy} + \sum{}''\left\{\frac{dH^{(1)}}{d\xi'}\frac{df}{d\eta'} - \frac{dH_*^{(1)}}{d\eta'}\frac{df}{d\xi'}\right\}. \quad (15)$$

The expanded portion of $\tilde{H}^{(m+1)}$ will then be

$$\Box' \check{S}^{(m)}. \quad (16)$$

It is easy to demonstrate that this part of $\tilde{H}^{(m+1)}$ contributed nothing to the function $H_*^{(m+1)}$, from which it follows that $H_*^{(m+1)}$ is not expanded at all. Evidently, the small divisor enters the function $S^{(m+1)}$ raised at most to the power ∇^{-2}. To demonstrate that $S^{(m+1)}$ is actually enlarged twice, it is sufficient to recall that the function (16) includes the terms

$$2H_{0,0,0}^{1,2,0}\frac{d\check{S}^{(m)}}{d\omega'} + 2H_{0,0,0}^{1,0,2}\frac{d\check{S}^{(m)}}{d\omega''}$$

(see p.201). The corresponding terms of $S^{(m+1)}$ are enlarged once more by the integration and, consequently, contain ∇^{-2}.

By treating in a similar manner the equation yielding $H_*^{(m+2)}$ and $S^{(m+2)}$, it will be found that the function $H_*^{(m+2)}$ contains at most ∇^{-1} and that the principal part of $S^{(m+2)}$ includes ∇^{-3}.

The general law is obvious. If $S^{(m)}$ $(m \geq 2)$ is the first expanded function, the series (4) of Section 25 will be expanded in powers of

$$\frac{\mu}{z - z_{\alpha,\beta}}.$$

Let us now pass to the types in which

$$q = 5.$$

Let us first assume that $S^{(3/2)}$ is the first function enlarged by the integration. According to Table 9 of Section 24, this will happen only in the cases /39

$$\alpha = 1, \qquad -5 \leq \beta \leq +4. \quad (17)$$

Then, the principal part of $S^{(3/2)}$ includes ∇^{-1}.

The equation yielding $H_*^{(2)}$ and $S^{(2)}$ is written as

$$DS^{(2)} = \tilde{H}^{(2)} - H_*^{(2)}.$$

The function $\tilde{H}^{(2)}$ of the second term does not contain the derivatives of the expanded function $S^{(3/2)}$. Thus, $H_*^{(2)}$ and $S^{(2)}$ present no small divisor at all.

The discussion can be continued as in the corresponding case for $q = 3$ (p.201), demonstrating that the most expanded terms of the functions $S^{(m)}$ and $H_*^{(m)}$ contain negative powers of ∇, as indicated in the Table on p.201, with the only exception that $S^{(2)}$ is not enlarged at all.

Thus, in the cases (17), the series (4) of Section 25 are expanded in powers of

$$\frac{\mu^{3/2}}{(z - \bar{z}_{\alpha,\beta})^2}.$$

Let us next assume that the first expanded function is $S^{(m)}$ $(m \geq 2)$. These cases are treated exactly like the corresponding cases for $q = 3$. The same results are obtained, with the only exception that $S^{(m+1/2)}$ is now not enlarged. Consequently, the series (4) of Section 25 progress in powers of

$$\frac{\mu}{z - \bar{z}_{\alpha,\beta}}.$$

Finally, a few remarks should be made on the types in which

$$q \geq 6.$$

Let $S^{(m)}$ be the first expanded function. We still have $m \geq 2$. As on /40
p.202, we demonstrate that the series (4) of Section 25 are then arranged in powers of

$$\frac{\mu}{z - \bar{z}_{\alpha,\beta}}.$$

After having studied the influence of the small divisors on the formal convergence of the series, we will further specify the definitions given at the beginning of Section 25. We will postulate that a planet is singly critical if the integration method of Section 25 is applicable; if, conversely, this method becomes illusory because of a small divisor $z_* - \bar{z}_{\alpha\beta}$, we will postulate that the planet is doubly critical.

In the cases

$$q = 3, \quad \alpha = 1, \quad \beta = -3, -2, -1, 0, +1, +2,$$

the planets cease to be singly critical as soon as the quantity $|z_* - \bar{z}_{\alpha\beta}|$ becomes comparable to $\mu^{1/4}$.

In the cases

$$q=3,\ \alpha=2,\ \beta=-5,-3,-1,+1,+3,$$
$$q=4,\ \alpha=1,\ \beta=-4,-3,-2,-1,0,+1,+2,+4,$$

this will happen as soon as $|z_* - \bar{z}_{\alpha\beta}|$ is of the order of $\mu^{1/2}$.

In the cases

$$q=3,\ \alpha=1,\ \beta=-8,-7,-6,-5,-4,+3,+4,+5,+6,+7,$$
$$q=3,\ \alpha=3,\ \beta=-8,-7,-5,-4,-2,-1,+1,+2,+4,+5,$$
$$q=5,\ \alpha=1,\ \beta=-5,-4,-3,-2,-1,0,+1,+2,+3,+4,$$

the limits between the singly and doubly critical planets are passed as soon as $|z_* - \bar{z}_{\alpha\beta}|$ is of the order of $\mu^{3/4}$.

Finally, in all the other cases, the planet ceases to be singly critical as soon as $|z_* - \bar{z}_{\alpha\beta}|$ becomes comparable to μ.

Section 27. /41

Sections 27 - 30 will be devoted to the theory of doubly critical planets. Thus, we assume that the initial value of the variable z is located in the vicinity of a certain value $\bar{z}_{\alpha\beta}$, defined at the beginning of Section 25.

The principles of Section 1 are still applicable. Only, it is now necessary to select the new characteristic function, denoted by H_{**}, in a different manner.

Let us return to eqs.(11) of Section 24. Starting from the known characteristic function $H(z, \xi', \xi''; \dot{y}, \eta', \eta'')$ and from the unknown function $H_{**}(z, \xi', \xi''; \dot{y}, \eta', \eta'')$, we must form the equation of partial derivatives

$$H\left(\frac{dS}{dy}, \frac{dS}{d\eta'}, \frac{dS}{d\eta''}; \dot{y}, \eta', \eta''\right) = H_{**}\left(z, \xi', \xi''; \frac{dS}{dz}, \frac{dS}{d\xi'}, \frac{dS}{d\xi''}\right). \tag{1}$$

There, we must introduce the expansions

$$\begin{aligned} H &= H^{(0)} + \mu^{1/2} H^{(1/2)} + \mu H^{(1)} + \mu^{3/2} H^{(3/2)} + \cdots, \\ H_{**} &= H_{**}^{(0)} + \mu^{1/2} H_{**}^{(1/2)} + \mu H_{**}^{(1)} + \mu^{3/2} H_{**}^{(3/2)} + \cdots, \\ S &= S^{(0)} + \mu^{1/2} S^{(1/2)} + \mu S^{(1)} + \mu^{3/2} S^{(3/2)} + \cdots \end{aligned} \tag{2}$$

and then compare the coefficients of the same powers of μ in the two members.

Setting

$$S^{(0)} = z\dot{y} + \xi'\eta' + \xi''\eta'', \quad S = S^{(0)} + S_1$$

eq.(1) becomes

$$H\left(z + \frac{dS_1}{d\dot{y}}, \xi' + \frac{dS_1}{d\eta'}, \xi'' + \frac{dS_1}{d\eta''}; \dot{y}, \eta', \eta''\right)$$
$$= H_{**}\left(z, \xi', \xi'': \dot{y} + \frac{dS_1}{dz}, \eta' + \frac{dS_1}{d\xi'}, \eta'' + \frac{dS_1}{d\xi''}\right). \tag{3}$$

This equation is satisfied for $\mu = 0$, if we put

$$H_{**}^{(0)} = H^{(0)} = h(z) - \sum{}'' \frac{\bar{\nu}'}{2}(\xi'^2 + \eta'^2). \tag{4}$$

By equating, in eq.(3), the coefficients of $\mu^{1/2}$ in the expansions of the two members, we obtain the equation /4

$$DS^{(1/2)} = H^{(1/2)} - H_{**}^{(1/2)}. \tag{5}$$

The second member is a finite trigonometric series with terms of the form

$$A \cos(\iota p\dot{y} + j'\omega' + j''\omega''). \tag{6}$$

The corresponding terms of $S^{(1/2)}$ then have the form

$$\frac{A \sin(\iota p\dot{y} + j'\omega' + j''\omega'')}{\iota p\nu(z) + (j' - j'')\bar{\nu}'}.$$

To obtain $S^{(1/2)}$ without small divisors, it is sufficient to combine, in $H_{**}^{(1/2)}$, all terms of $H^{(1/2)}$ in whose arguments we have

$$\iota = j\alpha, \quad j' - j'' = j\beta, \quad j = 0, \pm 1, \pm 2, \dots \tag{7}$$

Now, let f be any function periodic with respect to $\dot{y}$, ω', and ω'' and expandable in a trigonometric series with terms of the form of eq.(6). In Sections 27 and 28, we will denote by

$$[f]$$

the ensemble of the terms of f whose arguments correspond to the small divisors, i.e., the ensemble of terms of f where ι and $j' - j''$ satisfy one of the conditions (7).

In view of this, we must put in eq.(5):

$$H_{**}^{(1/2)} = [H^{(1/2)}].$$

Then, this equa ion (5) yields the function $S^{(1/2)}$ without small divisors.

Let us now compare the coefficients of μ in the two members of eq.(3). This yields

$$DS^{(1)} = \tilde{H}^{(1)} - H_{**}^{(1)}, \tag{8}$$

where we have used the notation /43

$$\tilde{H}^{(1)} = H^{(1)} + \sum{}'' \frac{dH^{(1/2)}}{d\xi'}\frac{dS^{(1/2)}}{d\eta'} + \frac{3}{2x_1^4}\left(\frac{dS^{(1/2)}}{d\dot{y}}\right)^2 - \sum{}'' \frac{\bar{\nu}'}{2}\left(\frac{dS^{(1/2)}}{d\eta'}\right)^2$$
$$- \frac{dH_{**}^{(1/2)}}{d\dot{y}}\frac{dS^{(1/2)}}{dz} - \sum{}'' \frac{dH_{**}^{(1/2)}}{d\eta'}\frac{dS^{(1/2)}}{d\xi'} + \sum{}'' \frac{\bar{\nu}'}{2}\left(\frac{dS^{(1/2)}}{d\xi'}\right)^2. \tag{9}$$

The function $H_{**}^{(1)}$ will be determined by the relation

$$H_{**}^{(1)} = [\tilde{H}^{(1)}].$$

Then, it is possible to solve eq.(8) for a function $S^{(1)}$ without small divisors.

Evidently, we can continue in this manner and successively determine all terms of the series (2).

It is of importance to mention that the odd powers of $\sqrt{\mu}$ vanish from the expansions yielding H, H_{**}, and S, as soon as q is an even number. This results from the last of the relations (25) in Section 24. We also mention here that the functions $H^{\left(\frac{q-2}{2}\right)}$, $H_{**}^{\left(\frac{q-2}{2}\right)}$, and $S^{\left(\frac{q-2}{2}\right)}$ are the first that can depend on the argument $\dot{y}$.

After having defined the functions $S(z, \xi', \xi''; \dot{y}, \eta', \eta'')$ and $H_{**}(z, \xi', \xi''; \dot{y}, \eta', \eta'')$, we start from the function

$$S(z_{**}, \xi'_{**}, \xi''_{**}; \dot{y}, \eta', \eta'')$$

and from the canonical transformation

$$z = \frac{dS}{d\dot{y}}, \qquad \dot{y}_{**} = \frac{dS}{dz_{**}},$$
$$\xi' = \frac{dS}{d\eta'}, \qquad \eta'_{**} = \frac{dS}{d\xi'_{**}},$$

$$\xi'' = \frac{dS}{d\eta''}, \qquad \eta''_{**} = \frac{dS}{d\xi''_{**}},$$

which can be also written as /44

$$
\begin{aligned}
z - z_{**} &= \frac{d(S - S^{(0)})}{d\dot{y}}, & \dot{y}_{**} - \dot{y} &= \frac{d(S - S^{(0)})}{dz_{**}}, \\
\xi' - \xi'_{**} &= \frac{d(S - S^{(0)})}{d\eta'}, & \eta'_{**} - \eta' &= \frac{d(S - S^{(0)})}{d\xi'_{**}}, \\
\xi'' - \xi''_{**} &= \frac{d(S - S^{(0)})}{d\eta''}, & \eta''_{**} - \eta'' &= \frac{d(S - S^{(0)})}{d\xi''_{**}}.
\end{aligned}
\tag{10}
$$

It is possible to express, by means of this transformation, the variables z, ξ', ξ''; $\dot{y}$, η', η'' as functions of the new variables z_{**}, ξ'_{**}, ξ''_{**}; $\dot{y}_{**}$, η'_{**}, η''_{**}. Thus, we obviously will have

$$H(z, \xi', \xi''; \dot{y}, \eta', \eta'') = H_{**}(z_{**}, \xi'_{**}, \xi''_{**}; \dot{y}_{**}, \eta'_{**}, \eta''_{**}).$$

Finally, the new variables satisfy the canonical system

$$
\begin{aligned}
\frac{dz_{**}}{dt_1} &= \frac{dH_{**}}{d\dot{y}_{**}}, & \frac{d\dot{y}_{**}}{dt_1} &= -\frac{dH_{**}}{dz_{**}}, \\
\frac{d\xi'_{**}}{dt_1} &= \frac{dH_{**}}{d\eta'_{**}}, & \frac{d\eta'_{**}}{dt_1} &= -\frac{dH_{**}}{d\xi'_{**}}, \\
\frac{d\xi''_{**}}{dt_1} &= \frac{dH_{**}}{d\eta''_{**}}, & \frac{d\eta''_{**}}{dt_1} &= -\frac{dH_{**}}{d\xi''_{**}}.
\end{aligned}
\tag{11}
$$

This system can be reduced to two degrees of freedom. To obtain this, let us define the quantities ρ'_{**}, ρ''_{**}; ω'_{**}, ω''_{**} by the relations

$$
\begin{aligned}
\xi'_{**} &= \rho'_{**} \cos \omega'_{**}, & \eta'_{**} &= \rho'_{**} \sin \omega'_{**}, \\
\xi''_{**} &= \rho''_{**} \cos \omega''_{**}, & \eta''_{**} &= \rho''_{**} \sin \omega''_{**}.
\end{aligned}
$$

In view of this, it is convenient to introduce first, in eq.(11), the canonical variables

$$
\begin{aligned}
&z_{**}, & &\dot{y}_{**}, \\
&\tfrac{1}{2}\rho'^2_{**}, & &\omega'_{**}, \\
&\tfrac{1}{2}\rho''^2_{**}, & &\omega''_{**},
\end{aligned}
$$

and, next, the new variables /45

$$\chi = \frac{z_{**} - z_{\alpha,\beta}}{\alpha p}, \qquad v = \alpha p y_{**} - \beta \omega''_{**},$$
$$\chi' = \tfrac{1}{2}\varrho'^2_{**}, \qquad v' = \omega'_{**} + \omega''_{**}, \tag{13}$$
$$\tfrac{1}{2}\varkappa_{**} = -\beta\chi + \tfrac{1}{2}\varrho'^2_{**} - \tfrac{1}{2}\varrho''^2_{**}, \qquad -\omega''_{**},$$

which obviously are also canonical. The function H_{**} is periodic with respect to the arguments y_{**}, ω'_{**}, and ω''_{**}, with the period 2π. In its trigonometric expansion, we encounter only the arguments

$$\iota p y_{**} + j'\omega'_{**} + j''\omega''_{**}$$

where

$$\iota = j\alpha, \quad j'' - j' = j\beta, \quad j = 0, \pm 1, \pm 2, \ldots$$

The general argument of the function H_{**} thus will be

$$\iota p y_{**} + j'\omega'_{**} + j''\omega''_{**} = jv + j'v'. \tag{14}$$

Consequently, the function H_{**} depends only on the two arguments v and v' and does not contain the argument $-\omega''_{**}$. The variables (13) satisfy the equations

$$\frac{d\chi}{dt_1} = \frac{dH_{**}}{dv}, \qquad \frac{dv}{dt_1} = -\frac{dH_{**}}{d\chi},$$
$$\frac{d\chi'}{dt_1} = \frac{dH_{**}}{dv'}, \qquad \frac{dv'}{dt_1} = -\frac{dH_{**}}{d\chi'}, \tag{15}$$

$$\varkappa_{**} = \text{const.}, \qquad \frac{d\omega''_{**}}{dt_1} = \frac{dH_{**}}{d\tfrac{1}{2}\varkappa_{**}}. \tag{16}$$

Equations (15) form a canonical system with two degrees of freedom. After its integration, we obtain the argument ω''_{**} by a quadrature, over the intermediary of the second equation of the system (16).

In the expansion of the characteristic function

$$H_{**}(z_{**}, \xi'_{**}, \xi''_{**};\ y_{**}, \eta'_{**}, \eta''_{**}),$$

which is given by the series /46

$$H_{**} = H^{(0)}_{**} + \mu^{1/2} H^{(1/2)}_{**} + \mu H^{(1)}_{**} + \cdots, \tag{17}$$

we must introduce

$$z_{**} = z_{\alpha,\beta} + \alpha p \chi,$$
$$\varrho'^2_{**} = 2\chi', \tag{18}$$
$$\varrho''^2_{**} = 2\chi' - 2\beta\chi - \varkappa_{**}$$

and also make use of eq.(14).

This will yield

$$H_{**}^{(m)} = \sum_{j,j'} h_{j,j'}^{(m)} \cos(jv + j'v'). \tag{19}$$

This sum includes only a finite number of terms. The quotients $h_{j,j'}^{(s)} : (\sqrt{\chi'})^{|j'|}$ are polynomials in χ' and x_{**} as well as functions rational with respect to χ. In addition, $h_{j,j'}^{(s)}$ remains finite for $\chi = 0$ since we have everywhere avoided the small integration divisors.

In particular, according to eqs.(4) and (18) and according to the definition of the quantity $z_{\alpha\beta}$ given at the beginning of Section 25, we will have

$$\begin{aligned} H_{**}^{(0)} &= h(z_{**}) - \frac{\bar{\nu}'}{2}(\varrho_{**}'^2 - \varrho_{**}''^2) \\ &= \frac{3}{2x_1^4}\alpha^2 p^2 \chi^2 - \{\alpha p \nu(\bar{z}_{\alpha,\beta}) + \beta\bar{\nu}'\}\chi + h(\bar{z}_{\alpha,\beta}) - \tfrac{1}{2}\bar{\nu}' x_{**} \\ &= A\chi^2 - \tfrac{1}{2}\bar{\nu}' x_{**} + \text{const.} \end{aligned}$$

with the notation

$$A = \frac{3\alpha^2 p^2}{2x_1^4}.$$

In the cases of interest here, the variable χ will have values near zero.

Section 28. /47

For the further discussion of the system (15) of the preceding Section, it becomes necessary to investigate also some of the other terms in the series (17) of the same Section. This will lead to an enormously large number of different cases, obtained in accordance with the values of the whole numbers q, α, and β.

For each given value of q, we will arrange the different cases (α, β) in various groups. For a given value of q, we will designate the group $(m)_q$ as the ensemble of the cases (α, β) for which $H_{**}^{(m)}$ is assumed as being the first coefficient dependent on v in the expansion of the characteristic function H_{**}. Below, we give the list of the first groups for the various values of q:

Groups $q = 3$:

$(\tfrac{1}{2})_3$: $H_{**}^{(1/2)} \neq 0$ dependent on v;

$(1)_3$: $H_{**}^{(1/2)} \equiv 0$, $H_{**}^{(1)}$ dependent on v;

$(\tfrac{3}{2})_3$: $H_{**}^{(1/2)} \equiv 0$, $H_{**}^{(1)}$ independent of v, $H_{**}^{(3/2)}$ dependent on v;

$(2)_3$: $H_{**}^{(1/2)} \equiv 0$, $H_{**}^{(1)}$ independent of v, $H_{**}^{(3/2)} \equiv 0$, $H_{**}^{(2)}$ dependent on v;

. . . .

$q = 4$:

$(1)_4$: $H_{**}^{(1)}$ dependent on v;

$(2)_4$: $H_{**}^{(1)}$ independent of v, $H_{**}^{(2)}$ dependent on v;

. . . .

$q = 5$:

$(1)_5$: $H_{**}^{(1)}$ independent of v, $H_{**}^{(3/2)}$ dependent on v;

$(2)_5$: $H_{**}^{(1)}$ independent of v, $H_{**}^{(3/2)} \equiv 0$, $H_{**}^{(2)}$ dependent on v;

. . . .

$q = 6$:

$(2)_6$: $H_{**}^{(1)}$ independent of v, $H_{**}^{(2)}$ dependent on v;

. . . .

. . . . .

We will then derive the different cases for each of these groups. /48

First, let q = 3.

We have shown that

$$H_{**}^{(1/2)} = [H^{(1/2)}].$$

The various terms of the function $H^{(1/2)}$ correspond to the six combinations in Table 5 of Section 24. In all arguments of this function, we have

$$\iota = 1, \quad -3 \leq j' + j'' \leq 0, \quad -3 \leq j' - j'' \leq +2. \tag{1}$$

To obtain the terms of the function $H_{**}^{(1/2)}$, we must retain, in $H^{(1/2)}$, all terms where

$$\iota = 1 = j\alpha, \quad j' - j'' = j\beta.$$

Thus, the group $(\frac{1}{2})_3$ includes the six cases

$$(\tfrac{1}{2})_3: \qquad \alpha = 1, \quad \beta = -3, -2, -1, 0, +1, +2.$$

Below, for the six cases, we give the expression of the function $H_{**}^{(1/2)}$ which always has the form of

$$H_{**}^{(1/2)} = h_{1,j'}^{(1/2)} \cos (v + j'v'):$$

$$\begin{array}{lll}
\alpha, & \beta: & H_{**}^{(1/2)}: \\
1, & -3: & 2H_{1,-3,0}^{1/2,3,0}(2\chi')^{3/2} \cos (v - 3v'), \\
1, & -2: & 2H_{1,-2,0}^{1/2,2,0}\, 2\chi' \cos (v - 2v'), \\
1, & -1: & 2H_{1,-1,0}^{1/2,1,0}(2\chi')^{1/2} \cos (v - v'), \\
1, & 0: & 2H_{1,0,0}^{1/2,0,0} \cos v, \\
1, & +1: & 2H_{1,-1,-2}^{1/2,1,2}(2\chi')^{1/2}(2\chi' - 2\chi - z_{**}) \cos (v - v'), \\
1, & +2: & 2H_{1,0,-2}^{1/2,0,2}(2\chi' - 4\chi - z_{**}) \cos v.
\end{array}$$

The coefficients $H_{1,j',j''}^{1/2,s,s'}$ are given by the general formula (23) of Section 24. The last of the relations in the system (25) of the same Section show that s = = 0 in this formula (23), from which it follows that the six mentioned coeffi- /49 cients are polynomials in e_0, with numerical coefficients, i.e., with constants.

Let us now pass to the function $H_{**}^{(1)}$. We will have

$$H_{**}^{(1)} = [\dot{\tilde{H}}^{(1)}].$$

The expression for the function $\tilde{H}^{(1)}$ is given by eq.(9) of the preceding Section. In all arguments of the derivatives of the functions $H^{(1/2)}$, $H_{**}^{(1/2)}$, and $S^{(1/2)}$ that enter this formula, the conditions (1) are satisfied. In all arguments of the squares of these derivatives as well as in all arguments of their pairwise products, we will thus have either

$$\iota = 0, \quad -3 \le j' + j'' \le +3, \quad -5 \le j' - j'' \le +5, \tag{2}$$

or else

$$\iota = 2, \quad -8 \le j' + j'' \le 0, \qquad -6 \le j' - j'' \le +4. \tag{3}$$

In addition, according to Tables 1 - 3 of Section 24, the relations (2) occur also for all arguments of the function $H^{(1)}$ which appears also in the expression of the function $\tilde{H}^{(1)}$. Thus, in all arguments of the functions $\tilde{H}^{(1)}$, $H_{**}^{(1)}$ and $S^{(1)}$, either the conditions (2) or the conditions (3) are realized.

To obtain the terms of the function $H_{**}^{(1)}$, it is necessary to retain, in $\tilde{H}^{(1)}$, first the terms in which

$$\iota = 0, \qquad j' = j'' = \text{even},$$

and then all terms in which

$$\iota = 2 = j\alpha, \qquad j' - j'' = j\beta.$$

Thus, in $H_{**}^{(1)}$, we will have in all cases a term which is a polynomial of the third degree in χ', rational in χ. In order to have, in $H_{**}^{(1)}$, terms that are dependent on v, it is necessary either that /50

$$\alpha = 1, \quad -3 \leq \beta \leq +2,$$

or else that

$$\alpha = 2, \quad -5 \leq \beta \leq +3, \quad (\beta \text{ being odd}).$$

Consequently, in the cases of the group $(1)_3$, the function $H_{**}^{(1)}$ has the form

$$H_{**}^{(1)} = h_{0,0}^{(1)} + \sum_{j'} h_{2,j'}^{(1)} \cos(2v + j'v'). \tag{4}$$

In addition, it is obvious that the group $(1)_3$ includes the cases

$(1)_3$ $$\alpha = 2, \quad \beta = -5, -3, -1, +1, +3.$$

In all cases of this group $(1)_3$, we find

$$H_{**}^{(1)} = h_{0,0}^{(1)} + \sum_{j'} h_{1,j'}^{(1)} \cos(v + j'v'). \tag{5}$$

Finally, in all other groups for q = 3, the expression of $H_{**}^{(1)}$ becomes

$$H_{**}^{(1)} = h_{0,0}^{(1)}. \tag{6}$$

In these formulas (4), (5), and (6), the quantity $h_{0,0}^{(1)}$ signifies a polynomial of the third degree in χ'. Nevertheless, the expression of $h_{0,0}^{(1)}$ may vary from one case to the other. The coefficients $h_{2,j}^{(1)}$ are polynomials in χ' of at most the third degree. Finally, the coefficients $h_{1,j}^{(1)}$ are odd polynomials in $\sqrt{\chi'}$ of at most the fifth degree. All these coefficients, in addition, are rational in χ and finite for $\chi = 0$.

We verified that the sum $\sum_{j'}$ in eq.(5) contains only a single term in the cases in which $\beta = -5, -3, +1, +3$ and only two terms in the case of $\beta = -1$.

We also would like to make the important statement that the coefficient of χ'^3 in the expression of $H_{**}^{(1)}$ is positive if q = 3. This results from the fact that the terms of the sixth degree of $H_{**}^{(1)}$ coincide with the terms of the sixth degree of the function /51

$$\frac{3}{2x_1^4}\left[\left(\frac{dS^{(0/2)}}{dy}\right)^2\right],$$

which is always positive if q = 3.

It is possible to continue the discussion without giving a detailed expression for the function $\widetilde{H}^{(3/2)}$. It is sufficient to recall the general character of this function. Here, we first have the term $H^{(3/2)}$ in whose arguments we have (according to Tables 5 and 6 of Section 24)

$$\iota = 1, \quad -4 \leq j' + j'' \leq +1, \quad -4 \leq j' - j'' \leq +5.$$

Each of the other terms of the function $\widetilde{H}^{(3/2)}$ can be considered as being composed of two factors. In the arguments of the first factor, the conditions (1) are satisfied; in the arguments of the second factor, either the relations (2) or else the relations (3) occur. In all these terms of the function $\widetilde{H}^{(3/2)}$ we will thus have either

$$\iota = 1, \quad -6 \leq j' + j'' \leq +3, \quad -8 \leq j' - j'' \leq +7,$$

or else

$$\iota = 3, \quad -9 \leq j' + j'' \leq 0, \quad -9 \leq j' - j'' \leq +6.$$

Thus, so that $H_{**}^{(3/2)}$ does not vanish identically, it is necessary either that

$$\alpha = 1, \quad -8 \leq \beta \leq +7,$$

or that

$$\alpha = 3, \quad -8 \leq \beta \leq +5$$

(α and β being mutually first).

Consequently, the cases of the group $\left(\frac{3}{2}\right)_3$ are as follows:

$$\left(\tfrac{3}{2}\right)_3: \quad \begin{array}{l} \alpha = 1, \quad \beta = -8, -7, -6, -5, -4, +3, +4, +5, +6, +7, \\ \alpha = 3, \quad \beta = -8, -7, -5, -4, -2, -1, +1, +2, +4, +5. \end{array}$$

In all these cases, the function $H_{**}^{(3/2)}$ has the form /52

$$H_{**}^{(3/2)} = \sum_{j'} h_{1,j'}^{(3/2)} \cos(v + j'v').$$

Passing, finally, to the function $\widetilde{H}^{(2)}$, it is easy to demonstrate that, in all its arguments, the numbers ι, j', and j'' satisfy the conditions contained in either one of the three following lines:

$$\begin{array}{lll} \iota = 0, & -6 \leq j' + j'' \leq +6, & -10 \leq j' - j'' \leq +10, \\ \iota = 2, & -9 \leq j' + j'' \leq +3, & -11 \leq j' - j'' \leq +9, \\ \iota = 4, & -12 \leq j' + j'' \leq 0, & -12 \leq j' - j'' \leq +8. \end{array}$$

Thus, in all cases, a part of the function $H^{(2)}_{**}$ has the form

$$h^{(2)}_{0,0} + h^{(2)}_{0,2} \cos 2v'.$$

This is the part which is independent of v. So that $H^{(2)}_{**}$ will contain terms dependent on v, it is necessary:

either that $\alpha = 1, \quad -5 \leq \beta \leq +4,$

or that $\alpha = 2, \quad -11 \leq \beta \leq +9$ (β being odd)

or else that $\alpha = 4, \quad -11 \leq \beta \leq +7$ (β being odd).

Consequently, the cases of the group $(2)_4$ are as follows:

$$(2)_4: \begin{array}{l} \alpha = 2, \quad \beta = -11, -9, -7, +5, +7, +9, \\ \alpha = 4, \quad \beta = -11, -9, -7, -5, -3, -1, +1, +3, +5, +7. \end{array}$$

In these particular cases, we obtain

$$H^{(2)}_{**} = h^{(2)}_{0,0} + h^{(2)}_{0,2} \cos 2v' + \sum_{j'} h^{(2)}_{1,j'} \cos (v + j'v').$$

We will now pass to the types in which q = 4. /53

The various terms of the function $H^{(1)}$ correspond to the combinations of the Tables 1 - 3 and 7 of Section 24. In any argument of this function, we will have either

$$\iota = 0, \quad -2 \leq j' + j'' \leq +2, \quad -4 \leq j' - j'' \leq +4, \tag{7}$$

or

$$\iota = 1, \quad -4 \leq j' + j'' \leq 0, \quad -4 \leq j' - j'' \leq +4, \neq +3. \tag{8}$$

To obtain $H^{(1)}_{**}$, it is necessary, in $H^{(1)}$, to retain first the terms for which

$$\iota = 0, \quad j' = j'' = \text{even},$$

and then the terms for which

$$\iota = 1 = j\alpha, \quad j' - j'' = j\beta.$$

Thus, the part of $H^{(1)}_{**}$ which is independent of v will be a polynomial in λ', independent of v'.

The cases of the group $(1)_4$ in which the function $H^{(1)}_{**}$ depends on the argument v, are as follows:

$$(1)_4: \quad \alpha = 1, \quad \beta = -4, -3, -2, -1, 0, +1, +2, +4.$$

In these cases, we have

$$H_{**}^{(1)} = h_{0,0}^{(1)} + \sum_{j'} h_{1,j'}^{(1)} \cos(v + j'v').$$

In all other cases, the expression for $H_{**}^{(1)}$ will be simply

$$H_{**}^{(1)} = h_{0,0}^{(1)}.$$

It is easy to write these expressions in detail. The first of the formulas in the system (22) of Section 24, for k = 1, as well as Tables 1 - 3 of the same Section, indicate that /54

$$\begin{aligned} h_{0,0}^{(1)} = {} & H_{0,0,0}^{1,4,0}\varrho_{**}'^2 + H_{0,0,0}^{1,2,2}\varrho_{**}'^2\varrho_{**}''^2 + H_{0,0,0}^{1,0,2}\varrho_{**}''^2 \\ & + H_{0,0,0}^{1,2,0}\varrho_{**}'^2 + H_{0,0,0}^{0,2}\varrho_{**}''^2 + H_{0,0,0}^{1,0,0} - \frac{2}{x_1^5} z^3{}_{**}. \end{aligned} \tag{9}$$

First, we must replace z in the expressions (23) of Section 24 by z_{**}, in order to obtain the coefficients $H_{0,0,0}^{1,2,0}$, $H_{0,0,0}^{1,0,2}$, and $H_{0,0,0}^{1,0,0}$; after this, we must make use of eqs.(18) of Section 27. In addition, according to Table 7 of Section 24, the expression of the part of $H_{**}^{(1)}$ which depends on v in the various cases of the group $(1)_4$ is given below:

$\alpha,$	$\beta:$	$H_{**}^{(1)} - h_{0,0}^{(1)}:$
1,	$-4:$	$2H_{1,-4,0}^{1,4,0}(2\chi')^2 \cos(v - 4v'),$
1,	$-3:$	$2H_{1,-3,0}^{1,3,0}(2\chi')^{3/2} \cos(v - 3v'),$
1,	$-2:$	$2H_{1,-2,0}^{1,2,0}\, 2\chi' \cos(v - 2v'),$
1,	$-1:$	$2H_{1,-1,0}^{1,1,0}(2\chi')^{1/2} \cos(v - v'),$
1,	$0:$	$2H_{1,0,0}^{1,0,0} \cos v + 2H_{1,-2,-2}^{1,2,2}\, 2\chi'(2\chi' - \varkappa_{**}) \cos(v - 2v'),$
1,	$+1:$	$2H_{1,-1,-2}^{1,1,2}(2\chi')^{1/2}(2\chi' - 2\chi - \varkappa_{**}) \cos(v - v'),$
1,	$+2:$	$2H_{1,0,-2}^{1,0,2}(2\chi' - 4\chi - \varkappa_{**}) \cos v,$
1,	$+4:$	$2H_{1,0,-4}^{1,0,4}(2\chi' - 8\chi - \varkappa_{**})^2 \cos v.$

Equation (23) of Section 24 and the last of the relations in the system (25) of the same Section indicate that the nine coefficients $H_{1,j',j''}^{1,s',s''}$ are polynomials in e_0 with numerical coefficients, i.e., are constants.

To study the function $H_{**}^{(2)}$ which is = $[\widetilde{H}^{(2)}]$, we note that $\widetilde{H}^{(2)}$ is obtained from eq.(9) of Section 27 if all the superscripts there are multiplied by 2.

Let us first set aside the first part $H^{(2)}$ of eq.(9), modified in this manner. Each of the other parts is a product of two factors (e.g., $\frac{dH^{(1)}}{d\xi'} \frac{dS^{(1)}}{d\eta'}$) in whose arguments the conditions of one or the other of the lines (7) and (8) are satisfied. In all arguments of each of these products, we consequently will have either /55

$$\iota = 0, \quad -4 \leq j' + j'' \leq +4, \quad -8 \leq j' - j'' \leq +8, \tag{10}$$

or

$$\iota = 1, \quad -6 \leq j' + j'' \leq +2, \quad -8 \leq j' - j'' \leq +8, \tag{11}$$

or else

$$\iota = 2, \quad -8 \leq j' + j'' \leq 0, \qquad -8 \leq j' - j'' \leq +8, \neq +7. \tag{12}$$

In the first part $H^{(2)}$ of the mentioned expression of $\widetilde{H}^{(2)}$, all arguments also satisfy the conditions of one or the other of the lines (10) and (11). This is indicated by Tables 1 - 4 and 7 - 8 of Section 24. Consequently, in all arguments of the function $\widetilde{H}^{(2)}$, the conditions of one or the other of the lines (10), (11), and (12) are satisfied.

To obtain $H^{(2)}_{**}$, it is sufficient to retain, in $\widetilde{H}^{(2)}$, the terms in whose arguments we have

$$\iota = j\alpha, \quad j' - j'' = j\beta, \quad j = 0, \pm 1, \pm 2, \ldots$$

In all cases, it will be found that the part independent of v has the form

$$h^{(2)}_{0,0} + h^{(2)}_{0,2} \cos 2v'.$$

Next, so that $H^{(2)}_{**}$ shall include terms dependent on v, it is necessary either that

$$\alpha = 1, \qquad -8 \leq \beta \leq +8,$$

or that

$$\alpha = 2, \qquad -7 \leq \beta \leq +5 \qquad (\beta \text{ being odd})$$

Thus, the group $(2)_4$ is composed of the cases

$$(2)_4: \begin{array}{l} \alpha = 1, \quad \beta = -8, -7, -6, -5, +3, +5, +6, +7, +8, \\ \alpha = 2, \quad \beta = -7, -5, -3, -1, +1, +3, +5. \end{array}$$

In these cases, except in the case in which $\alpha = 1$ and $\beta = +3$, we will have /56

$$H_{**}^{(2)} = h_{0,0}^{(2)} + h_{0,2}^{(2)} \cos 2v' + \sum_{j'} h_{1,j'}^{(2)} \cos (v + j'v').$$

In the case in which $\alpha = 1$ and $\beta = +3$, a term is added for which $j = 2$; we found that we there have $j' = 0$, such that the term in question has the form

$$h_{2,0}^{(2)} \cos 2v.$$

Let us now assume that $q = 5$.

Then, we have

$$H^{(3/2)} \equiv 0, \quad H_{**}^{(3/2)} \equiv 0, \quad S^{(3/2)} \equiv 0.$$

It follows from this that

$$H_{**}^{(1)} = [H_{*}^{(1)}].$$

In all arguments of the function $H^{(1)}$, we have

$$\iota = 0, \quad -2 \leq j' + j'' \leq +2, \quad -4 \leq j' - j'' \leq +4. \tag{13}$$

Consequently, the function $H_{**}^{(1)}$ will in all cases be the ensemble of the terms of $H^{(1)}$, where $j' = j'' = 0$. Thus, we always have

$$H_{**}^{(1)} = h_{0,0}^{(1)}.$$

The expression of $h_{0,0}^{(1)}$ will also be given by eq.(9).

We now pass to the function $H_{**}^{(3/2)}$ which is given by the formula

$$H_{**}^{(3/2)} = [H^{(3/2)}].$$

In all arguments of the function $H^{(3/2)}$, we have, in accordance with Table 9 of Section 24,

$$\iota = 1, \quad -5 \leq j' + j'' \leq 0, \quad -5 \leq j' - j'' \leq +4. \tag{14}$$

Consequently, the function $H_{**}^{(3/2)}$ vanishes identically except in the cases

$$\left(\tfrac{3}{2}\right)_5: \ \alpha = 1, \ \beta = -5, -4, -3, -2, -1, 0, +1, +2, +3, +4,$$ /57

which thus constitute the group $\left(\frac{3}{2}\right)_5$. In these cases, we have

$$H_{**}^{(3/2)} = \sum_{j'} h_{1,j'}^{(3/2)} \cos (v + j'v').$$

Ordinarily, the sum comprises a single term; only in the cases in which $\beta = -1$ or 0 will it include two terms.

The function $H^{(2)}_{**}$ will have the expression

$$H^{(2)}_{**} = [\tilde{H}^{(2)}], \tag{15}$$

where

$$\tilde{H}^{(2)} = H^{(2)} + \sum{}' \left\{ \frac{dH^{(1)}}{d\xi'} \frac{dS^{(1)}}{d\eta'} - \frac{dH^{(1)}_{**}}{d\eta'} \frac{dS^{(1)}}{d\xi'} \right\} + \sum{}'' \frac{\nu'}{2} \left\{ \left(\frac{dS^{(1)}}{d\xi'} \right)^2 - \left(\frac{dS^{(1)}}{d\eta'} \right)^2 \right\}. \tag{16}$$

In all arguments of the derivatives of the functions $H^{(1)}$, $H^{(1)}_{**}$, and $S^{(1)}$, the relations (13) are satisfied. It follows from this that

$$\iota = 0, \quad -4 \le j' + j'' \le +4, \quad -8 \le j' - j'' \le +8 \tag{17}$$

in all arguments of the function $\tilde{H}^{(2)} - H^{(2)}$. In accordance with Table 4 of Section 24, the same relations are fulfilled for all arguments of the function $H^{(2)}$. Consequently, they are satisfied by the arguments of $\tilde{H}^{(2)}$

Thus, the function $H^{(2)}_{**}$, in all cases, is independent of v. This means that the group $(2)_5$ does not exist.

In the expression of the function $H^{(2)}_{**}$, we could expect a term in cos $2v'$; however, this term vanishes identically, and the function $H^{(2)}_{**}$ will still have the form

$$H^{(2)}_{**} = h^{(2)}_{0,0},$$

where $h^{(2)}_{0,0}$ is a polynomial in X' of the third degree and independent of v as well as of v'.

Let us finally pass to the types in which q = 6. /58

Our above statements on the function $H^{(1)}_{**}$ for the case in which q = 5, remain valid also for q ≥ 5.

The functions $H^{(2)}_{**}$ and $\tilde{H}^{(2)}$ are also given by eqs (15) and (16). The conditions (17) are fulfilled in all arguments of the function $\tilde{H}^{(2)} - H^{(2)}$ and also in those of the arguments of $H^{(2)}$ in which $\iota = 0$. In the other arguments of this function $H^{(2)}$, we will have, in accordance with Table 10 of Section 24,

$$\iota = 1, \quad -0 \le j' + j'' \le 0, \quad -6 \le j' - j'' \le +6, \quad \nu + 5.$$

As for the case of q = 5, we find also for q = 6 that the part of $H^{(2)}_{**}$, which is independent of v, is also independent of v'. For all values of α and β, this part of $H^{(2)}_{**}$ is a polynomial of the third degree in χ', rational in χ.

The function $H^{(2)}_{**}$ might also include a part dependent on v, but this will happen only in the cases of the group $(2)_6$:

$$(2)_6: \quad \begin{aligned} &\alpha = 1, \\ &\beta = -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +6. \end{aligned}$$

In these cases, we will have

$$H^{(2)}_{**} = h^{(2)}_{0,0} + \sum_{j'} h^{(2)}_{1,j'} \cos (v + j'v').$$

According to Table 10 of Section 24, the sum includes a single term if $\beta = j' - j'' = -6, -5, -4, -3, +1, +3, +4, +6$ and only two terms in the cases in which $\beta = -2, -1, 0, +2$.

In all other cases, we obtain

$$H^{(2)}_{**} = h^{(2)}_{0,0}.$$

We should like to make one more remark on the types in which $q > 6$.

In that case, the functions $H^{(1)}_{**}$ and $H^{(2)}_{**}$ are polynomials in χ' of the second resp. third degree, rational in χ and independent of v and of v'. /59

Section 29.

We know of no completely general method for performing the formal integration of eqs.(15) of Section 27. The canonical system in question enters the type studied in Section 1. However, the method of reduction discussed there is not directly applicable to the present case, since the derivative of $H^{(0)}_{**}$ with respect to χ is assumed as small and since $H^{(0)}_{**}$ does not depend on the variable χ'. Nevertheless, we will investigate here a certain number of rather extensive cases in which a formal integration of the investigated system is well possible.

Let us assume first that the initial value of χ is of the order of $\mu^{1/2}$ and, in addition, that the initial value of χ' is neither too large nor too small.

In our investigation, it will first be necessary to exclude the cases of the groups $(\frac{1}{2})_3$, $(1)_3$, and $(1)_4$, and treat them separately later.

It is convenient to put

$$\chi = \mu^{1/2}\chi_1, \qquad \chi' = \chi^{r} + \mu^{1/2}\chi'_1, \tag{1}$$

$$H_{**} + \frac{\bar{\nu}'}{2}\varkappa_{**} - C = \mu G, \quad \tau = \mu^{1/2} t_1 = \mu^{1/2} t.$$

We will assume that the initial value of $|\chi_1|$ is comparable in magnitude to unity and that the value of the positive parameter $\overline{\chi'}$ is sufficiently large so that $\sqrt{\overline{\chi'}}$ can be expanded in powers of $\mu^{1/2}\chi_1'$.

The variables χ_1, χ_1'; v, v' satisfy the equations

$$
\begin{aligned}
\frac{d\chi_1}{d\tau} &= \frac{dG}{dv}, & \frac{dv}{d\tau} &= -\frac{dG}{d\chi_1}, \\
\frac{d\chi'_1}{d\tau} &= \frac{dG}{dv'}, & \frac{dv'}{d\tau} &= -\frac{dG}{d\chi'_1}.
\end{aligned} \tag{2}
$$

In several formulas of the present Section, we will use the notation /60

$$\overline{f}$$

to indicate the value of any function $f(\chi, \chi')$ for the values $\chi = 0$, $\chi' = \overline{\chi'}$ of the two variables χ and χ'.

In view of this, we obviously will have

$$
\begin{aligned}
G &= G_0 + \mu^{1/2} G_{1/2} + \mu G_1 + \cdots, \\
G_0 &= A\chi_1^2 + \overline{h_{0,0}^{(1)}}, \\
G_{1/2} &= \frac{\overline{dh_{0,0}^{(1)}}}{d\chi}\chi_1 + \frac{\overline{dh_{0,0}^{(1)}}}{d\chi'}\chi_1' + \overline{H_{**}^{(1/2)}}, \\
G_1 &= \frac{1}{2}\frac{\overline{d^2h_{0,0}^{(1)}}}{d\chi^2}\chi_1^2 + \frac{\overline{d^2h_{0,0}^{(1)}}}{d\chi\, d\chi'}\chi_1\chi_1' + \frac{1}{2}\frac{\overline{d^2h_{0,0}^{(1)}}}{d\chi'^2}\chi_1'^2 \\
&\quad + \frac{\overline{dH_{**}^{(1/2)}}}{d\chi}\chi_1 + \frac{\overline{dH_{**}^{(1/2)}}}{d\chi'}\chi_1' + \overline{H_{**}^{(2)}}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;
\end{aligned}
$$

In a given case of the group $(r)_q$, the quantity G_{r-1} will be the first of the above functions that depend on v.

Since the initial value of $|\chi_1|$ is comparable in magnitude to unity, we can reduce the system (2) to another system with one degree of freedom. According to the method given in Section 1, we will start from the equation

$$G\left(\frac{dS}{dv}, \frac{dS}{dv'}; v, v'\right) = G^*\left(\chi_1, \chi_1'; \frac{dS}{d\chi_1'}\right)$$

by introducing there

$$G^* = G_0^* + \mu^{1/2} G^*_{1/2} + \mu G_1^* + \cdots,$$
$$S = S_0 + \mu^{1/2} S_{1/2} + \mu S_1 + \cdots. \tag{3}$$

Putting

$$S_0 = \chi_1 v + \chi_1' v', \quad S = S_0 + \delta S,$$

the equation in S can be written as /61

$$G\left(\chi_1 + \frac{d\delta S}{dv}, \chi_1' + \frac{d\delta S}{dv'}; v, v'\right) = G^*\left(\chi_1, \chi_1'; v' + \frac{d\delta S}{d\chi_1'}\right).$$

In the expansions of the two members of this equation, we must compare the coefficients of the same powers of $\mu^{1/2}$.

The notation

$$[f],$$

is to represent, in this Section 29, the mean value of any function f, periodic in v.

In view of this, we find successively

$$G_0^* = G_0 = A\chi_1^2 + \overline{h^{(1)}_{0,0}};$$

$$2A\chi_1 \frac{dS_{1/2}}{dv} + G_{1/2} = G^*_{1/2},$$

$$\therefore\ G^*_{1/2} = \frac{\overline{dh^{(1)}_{0,0}}}{d\chi}\chi_1 + \frac{\overline{dh^{(1)}_{0,0}}}{d\chi'}\chi_1',$$

$$S_{1/2} = -\frac{1}{2A\chi_1}\int \overline{H^{(1/2)}_{**}}\,dv;$$

$$2A\chi_1 \frac{dS_1}{dv} + A\left(\frac{dS_{1/2}}{dv}\right)^2 + \frac{dG_{1/2}}{d\chi_1}\frac{dS_{1/2}}{dv} + \frac{dG_{1/2}}{d\chi_1'}\frac{dS_{1/2}}{dv'} + G_1 = G_1^*,$$

$$\therefore\ G_1^* = [G_1] + \frac{1}{4A\chi_1^2}[(\overline{H^{(1/2)}_{**}})^2]$$
$$= \frac{1}{2}\frac{\overline{d^2h^{(1)}_{0,0}}}{d\chi^2}\chi_1^2 + \frac{\overline{d^2h^{(1)}_{0,0}}}{d\chi\, d\chi'}\chi_1\chi_1' + \frac{1}{2}\frac{\overline{d^2h^{(1)}_{0,0}}}{d\chi'^2}\chi_1'^2$$
$$+ [\overline{H^{(2)}_{**}}] + \frac{1}{4A\chi_1^2}[(\overline{H^{(1/2)}_{**}})^2];$$

. . . .

In any case of the group $(r)_q$, the quantity S_{r-1} obviously is the first of the functions $S_{1/2}$, S_1, ... which is not identically zero.

As soon as the functions G^* and S are known, we can start out from the function /62

$$S(\chi_1^*, \chi_1'^*; v, v')$$

to form the canonical transformation

$$\chi_1 - \chi_1^* = \frac{d(S-S_0)}{dv}, \quad v^* - v = \frac{d(S-S_0)}{d\chi_1^*},$$
$$\chi_1' - \chi_1'^* = \frac{d(S-S_0)}{dv'}, \quad v'^* - v' = \frac{d(S-S_0)}{d\chi_1'^*}. \tag{4}$$

In virtue of this transformation, we will have

$$G(\chi_1, \chi_1'; v, v') = G^*(\chi_1^*, \chi_1'^*; v'^*).$$

The new variables χ_1^*, $\chi_1'^*$, v^*, v'^* satisfy the equations

$$\frac{d\chi_1^*}{d\tau} = 0, \qquad \frac{dv^*}{d\tau} = -\frac{dG^*}{d\chi_1^*}, \tag{5}$$

$$\frac{d\chi_1'^*}{d\tau} = \frac{dG^*}{dv'^*}, \qquad \frac{dv'^*}{d\tau} = -\frac{dG^*}{d\chi_1'^*}. \tag{6}$$

Thus, χ_1^* is a constant. Let us assume that $|\chi_1^*|$ is comparable in magnitude to unity.

The system (6) is canonical, with one degree of freedom. By means of the first integral

$$G^*(\chi_1^*, \chi_1'^*; v'^*) = g, \tag{7}$$

where g is an arbitrary constant, the quantity $\chi_1'^*$ will be given as a function of the angular variable v'^*. We postulate that the variations of $\chi_1'^*$ are comparable in magnitude to unity or less. This results from the expressions of the functions G_0^*, $G_{1/2}^*$, G_1^*. In fact, the first of these functions is constant with χ_1^*; in the two other functions, the three coefficients /63

$$\frac{dh_{0,0}^{(1)}}{d\chi'}, \quad \frac{d^2h_{0,0}^{(1)}}{d\chi d\chi'}, \quad \frac{d^2h_{0,0}^{(1)}}{d\chi'^2}$$

are not all small, since they only include two arbitrary parameters $\varkappa_{**}$ and $\bar{\chi}'$.

In view of this, the system (6) will yield $\chi_1'^*$, $\cos v'^*$, and $\sin v'^*$ as periodic functions of t, with a period Π which is extremely long, at least comparable to μ^{-2}. The absolute value of the function $\chi_1'^*$, defined in this manner, always remains inferior to a fixed quantity comparable in magnitude to unity. With respect to the variations in the angle v'^*, two cases must be differentiated, depending on the values of the constant g. Either the argument v'^* will have a mean motion such that v'^* increases or decreases by 2π during the period or else this argument will execute periodic oscillations between two limits. The second case is that of libration.

After integration of the system (6), the second of the equations in the system (5) will yield v^*, composed of a linear part in t and of a periodic part with the period Π. The principal motion of the argument v^* is comparable in magnitude to $\mu^{3/2}$.

Let us now return to the relations (4). The two equations on the right-hand side are of the generalized Lagrange type with two variables. After solving these equations, we obtain $v - v^*$, $v' - v'^*$, $\chi_1 - \chi_1^*$, $\chi_1' - \chi_1'^*$ as functions of χ_1^*, $\chi_1'^*$, v^*, and v'^*, periodic in v^* and v'^*, with the period 2π. These differences, incidentally, are of the order of μ^{r-1} in the cases of the group $(r)_q$.

We have assumed that $r \geq \frac{3}{2}$, from which it follows that the mentioned differences are small quantities, comparable to $\mu^{1/2}$ or smaller. On replacing χ'^*, v^* and v'^* by their expressions as functions of t, we finally obtain $v - v^*$, $v' - v'^*$, χ_1, and χ_1' as oscillating and finite functions of t. Thus, the formal integration of eqs.(15) of Section 27 is possible if the initial value of χ is of the order of $\mu^{1/2}$, except possibly in the cases of the excluded groups $(\frac{1}{2})_3$, $(1)_3$, and $(1)_4$.

Below, we give an important result of the above statements. Let us consider the system (15) of Section 27, by naturally assuming that the functions $H_{**}^{(1/2)}$ and $H_{**}^{(1)}$ do not depend on v. Let us consider a general solution in which, at the origin of time, the absolute value of the variable χ is comparable in magnitude to $\mu^{1/2}$, whereas the corresponding value of the positive variable χ' is neither too large nor too small. Then, the variable χ always remains more or less constant and of the order of $\mu^{1/2}$, while the variations of χ' are of the order of $\mu^{1/2}$. Later, we will have occasion to make use of this result, in studying the stability of motion of doubly critical planets from the viewpoint of formal calculus. /64

Here, we will assume that the initial value of χ is of the order of μ or smaller and also that the initial value of χ' is neither too large nor too small. First, we will exclude the cases in which the functions $H_{**}^{(1/2)}$, $H_{**}^{(1)}$, $H_{**}^{(3/2)}$, and $H_{**}^{(2)}$ are not all independent of v, i.e., the cases of the groups $(\frac{1}{2})_3$, $(1)_3$, $(\frac{3}{2})_3$, $(2)_3$; $(1)_4$, $(2)_4$; $(\frac{3}{2})_5$, $(2)_5$ and $(2)_6$.

We can set

$$\chi = \mu\chi_1, \qquad \chi' = \bar{\chi}' + \mu\chi_1', \tag{8}$$

$$H_{**} + \frac{\bar{\nu}'}{2}\varkappa_{**} - \text{const.} = \mu^2 G, \quad \tau = \mu t_1 = \mu^2 t.$$

Let us assume that the value of the positive parameter χ' is sufficiently large so that $\sqrt{\chi'}$ can be expanded in powers of $\mu\chi_1'$.

This will lead to the equations

$$\frac{d\chi_1}{d\tau} = \frac{dG}{dv}, \quad \frac{dv}{d\tau} = -\frac{dG}{d\chi_1},$$
$$\frac{d\chi_1'}{d\tau} = \frac{dG}{dv'}, \quad \frac{dv'}{d\tau} = -\frac{dG}{d\chi_1'}. \tag{9}$$

The function G can be expanded in powers of $\sqrt{\mu}$:

$$G = G_0 + \mu^{1/2} G_{1/2} + \mu G_1 + \cdots. \tag{10}$$

In accordance with Section 28, we know that

$$H_{**}^{(1/2)} = 0, \quad H_{**}^{(1)} = h_{0,0}^{(1)},$$ /65
$$H_{**}^{(3/2)} = 0, \quad H_{**}^{(2)} = h_{0,0}^{(2)} + h_{0,2}^{(2)} \cos 2v'.$$

In view of this, the function G_0 is expressed by

$$G_0 = A\chi_1^2 + \frac{d\overline{h_{0,0}^{(1)}}}{d\chi}\chi_1 + \frac{d\overline{h_{0,0}^{(1)}}}{d\chi'}\chi_1' + \overline{h_{0,2}^{(2)}} \cos 2v'.$$

In Section 28, we demonstrated that

$$h_{0,2}^{(2)} \equiv 0, \quad if \quad q \geq 5.$$

Thus, as soon as $q \geq 5$, the system (9) will have the normal form given in Section 1 and can be integrated from the viewpoint of formal calculus, if the two quantities

$$2A\chi_1 + \frac{d\overline{h_{0,0}^{(1)}}}{d\chi} \quad and \quad \frac{d\overline{h_{0,0}^{(1)}}}{d\chi'} \tag{11}$$

are not both small.

However, we also must consider the types in which $q = 3$ or 4. So as not to go into too much detail, we will assume that the value of the derivative

$$\frac{d\overline{h_{0,0}^{(1)}}}{d\chi'}$$

is not too close to zero. It is then easy to reduce the system (9) to another system which has the normal form. It is sufficient to set

$$\chi_1'' = \chi_1' + \frac{\overline{h^{(2)}_{0,2}}}{\frac{d\overline{h^{(1)}_{0,0}}}{d\chi'}} \cos 2v'$$

and to use the canonical variables

$$\chi_1, \ \chi_1''; \ v, \ v'. \tag{12}$$

These variables satisfy the equations /66

$$\begin{aligned} \frac{d\chi_1}{d\tau} &= \frac{dG}{dv}, \quad \frac{dv}{d\tau} = -\frac{dG}{d\chi_1}, \\ \frac{d\chi_1''}{d\tau} &= \frac{dG}{dv'}, \quad \frac{dv'}{d\tau} = -\frac{dG}{d\chi_1''} \end{aligned} \tag{13}$$

where G is now expressed as a function of the variables (12). The principal part of G will then be

$$G_0 = A\chi_1^2 + \frac{d\overline{h^{(1)}_{0,0}}}{d\chi}\chi_1 + \frac{d\overline{h^{(1)}_{0,0}}}{d\chi'}\chi_1'.$$

Thus, the canonical system is reduced again to the normal form given in Section 1. It is possible to integrate this system, since we have assumed that the coefficient of χ_1' in the above expression of G_0 is not small.

Here, we will make a final remark on the variation of the angular variables in the cases with which we are concerned since p.224. If the two quantities (11) are not approximately at a commensurable and simple ratio, then each linear combination of the arguments v and v' will certainly include a term linear in t. If, conversely, a divisor of the form of

$$k\left(2A\chi_1 + \frac{d\overline{h^{(1)}_{0,0}}}{d\chi}\right) + k'\frac{d\overline{h^{(1)}_{0,0}}}{d\chi'}$$

(where k and k' are two integers) becomes small and if, in addition, the integration constants satisfy certain inequalities, it will happen that the linear combination

$$kv + k'v' = k(\alpha p\dot{y}_{**} - \beta\omega''_{**}) + k'(\omega'_{**} + \omega''_{**}) \tag{14}$$

of the arguments remains enclosed between two limits. Let us see what this means. In the Introduction to Part II of this research, we defined what we called the eccentric vector of a minor planet as well as the longitude of this eccentric vector. Let now n, n', and $-\mu\nu''$ be the mean motions defined on p.196. Let, similarly, $-\mu\nu'$ be the mean motion of the longitude of the eccentric vector. In view of this, it is obvious that the following relation will exist between the mean motions: /67

$$\alpha k\{pn-(p+q)n'\}+k'\mu\nu'+(k'-k\beta)\mu\nu''=0$$

in the case of libration.

Section 30.

We will now resume the integration of the system (15) of Section 27, for certain of the cases that we had excluded in the preceding Sections.

Consider the cases of the groups $(r)_q$ where $r = \frac{1}{2}$, 1, or $\frac{3}{2}$. It is here a question of integrating the mentioned system, assuming that the initial value of the unknown χ is of the order of $\mu^{r/2}$ or smaller and, in addition, that the initial value of χ' is neither too large nor too small.

We then set

$$\chi=\mu^{r/2}\chi_1,\qquad \chi'=\chi'+\mu^{r/2}\chi_1',$$
$$H_{**}+\frac{\nu'}{2}\chi_{**}-C=\mu^r G,\quad \tau=\mu^{r/2}t_1=\mu^{r/2+1}t. \tag{1}$$

The variables χ_1, χ_1'; v, v' satisfy the canonical system

$$\frac{d\chi_1}{d\tau}=\frac{dG}{dv},\quad \frac{dv}{d\tau}=-\frac{dG}{d\chi_1},$$
$$\frac{d\chi_1'}{d\tau}=\frac{dG}{dv'},\quad \frac{dv'}{d\tau}=-\frac{dG}{d\chi_1'}. \tag{2}$$

In the cases of the group $(\frac{1}{2})_3$, the function G can be expanded in powers of $\mu^{1/4}$. We set

$$G=G_0+\mu^{1/4}G_{1/4}+\mu^{1/2}G_{1/2}+\cdots. \tag{3}$$

The first functions G_k then are expressed by /68

$$G_0=A\chi_1^2+H_{**}^{(1/2)}=A\chi_1^2+\bar{h}_{1,j'}^{(1/2)}\cos(v+j'v'),$$
$$G_{1/4}=\frac{dH_{**}^{(1/2)}}{d\chi}\chi_1+\frac{dH_{**}^{(1/2)}}{d\chi'}\chi_1', \tag{4}$$

$$G_{1/2} = \frac{1}{2}\frac{d^2\overline{H_{**}^{(1/2)}}}{d\chi^2}\chi_1^2 + \frac{d^2\overline{H_{**}^{(1/2)}}}{d\chi\, d\chi'}\chi_1\chi_1' + \frac{1}{2}\frac{d^2\overline{H_{**}^{(1/2)}}}{d\chi'^2}\chi_1'^2 + \overline{H_{**}^{(1)}}.$$

In the cases of the groups $(1)_3$ and $(1)_4$, the function G can be expanded in powers of $\mu^{1/2}$. Then, let

$$G = G_0 + \mu^{1/2}G_{1/2} + \mu G_1 + \cdots. \tag{5}$$

In these cases, we will have

$$\begin{aligned} G_0 &= A\chi_1^2 + \overline{H_{**}^{(1)}} = A\chi_1^2 + \overline{h_{0,0}^{(1)}} + \sum_{j'} \overline{h_{1,j'}^{(1)}} \cos(v + j'v'), \\ G_{1/2} &= \frac{d\overline{H_{**}^{(1)}}}{d\chi}\chi_1 + \frac{d\overline{H_{**}^{(1)}}}{d\chi'}\chi_1' + \overline{H_{**}^{(3/2)}}. \end{aligned} \tag{6}$$

Finally, in the cases of the groups $(\frac{3}{2})_3$ and $(\frac{3}{2})_5$, the expansion of G will have the form

$$G = G_0 + \mu^{1/4}G_{1/4} + \mu^{1/2}G_{1/2} + \cdots. \tag{7}$$

Here, we will have

$$\begin{aligned} G_0 &= A\chi_1^2 + \overline{H_{**}^{(3/2)}} = A\chi_1^2 + \sum_{j'} \overline{h_{1,j'}^{(3/2)}} \cos(v + j'v'), \\ G_{1/4} &= \frac{d\overline{H_{**}^{(1)}}}{d\chi}\chi_1 + \frac{d\overline{H_{**}^{(1)}}}{d\chi'}\chi_1' = \frac{d\overline{h_{0,0}^{(1)}}}{d\chi}\chi_1 + \frac{d\overline{h_{0,0}^{(1)}}}{d\chi'}\chi_1'. \end{aligned} \tag{8}$$

In all the investigated cases, the functions G_m are polynomials in χ_1 and χ_1', periodic with respect to the arguments v and v', with the period 2π.

The system (2) does not have the normal form investigated in Section 1 since the principal part of the characteristic function depends on the angular variables.

We have shown in Section 28 that the function $H_{**}^{(1/2)}$ in all cases of the group $(\frac{1}{2})_3$ includes a single term which, in addition, is periodic. We also showed there that the function $H_{**}^{(1)}$ includes a single periodic term in all cases of the groups $(1)_3$ and $(1)_4$, except in the cases 69

$$\begin{aligned} q &= 3, \quad \alpha = 2, \quad \beta = -1, \\ q &= 4, \quad \alpha = 1, \quad \beta = 0, \end{aligned} \tag{9}$$

where it contains two terms. Similarly, in several cases of the groups $(\frac{3}{2})_3$ and $(\frac{3}{2})_5$, the function $H_{**}^{(3/2)}$ contains a single periodic term; in other cases which we do not believe necessary to specify here, the function $H_{**}^{(3/2)}$ includes two or three periodic terms.

In the discussion given below, we will exclude the cases in which the function G_0 contains more than one periodic term. The cases, excluded in this manner, actually seem to present serious difficulties. In attempting to treat these cases, one encounters differential equations which it seems impossible to integrate by the presently known analytic methods.

If only a single periodic term exists in G_0, it is easy to reduce the canonical system (2) to an especially interesting form, by a very simple canonical transformation:

$$x = \chi_1, \qquad y = v + j'v',$$
$$x' = -j'\chi_1 + \chi_1' \quad y' = v'. \tag{13}$$

(The variable y, temporarily used here, must not be confused with the argument y defined in Section 16.) The variables x, x'; y, y' satisfy a canonical system of the type

$$\frac{dx}{d\tau} = \frac{dF}{dy}, \quad \frac{dy}{d\tau} = -\frac{dF}{dx},$$
$$\frac{dx'}{d\tau} = \frac{dF}{dy'}, \quad \frac{dy'}{d\tau} = -\frac{dF}{dx'}, \tag{14}$$

whose characteristic function can be expanded in powers of a small parameter /70
μ' ($= \mu^{1/4}$ or $\mu^{1/2}$):

$$F = F^{(0)} + \mu' F^{(1)} + \mu'^2 F^{(2)} + \cdots. \tag{15}$$

The principal term of this development has the following special form:

$$F^{(0)} = Ax'^2 - B\cos y, \tag{16}$$

where B is a constant, which we obviously can assume to be positive without restricting the generality. The following functions $F^{(1)}$, $F^{(2)}$, ... are polynomials in x and x' and are periodic with respect to y and y', having the period 2π. In the actual cases, the functions $F^{(0)}$ and $F^{(1)}$ are directly derived from eqs.(4), (6), or (8).

We will now investigate the system (14).

This system is easy to integrate if $\mu' = 0$. Then the Jacobi equation of partial derivatives becomes

$$A\left(\frac{dS}{dy}\right)^2 - B\cos y = B\gamma,$$

where γ is a parameter. This equation is satisfied by the function

$$S = \sqrt{\frac{B}{A}}\int\sqrt{\gamma + \cos y}\,dy. \tag{17}$$

Let us now assume that the small parameter μ' is no longer zero. In eq.(17), we can consider γ as a certain function of a variable x^0 and replace x and y by new variables x^0 and y^0, by means of the canonical transformation

$$\begin{aligned} x &= \frac{dS}{dy} = \sqrt{\frac{B}{A}}\sqrt{\gamma + \cos y}, \\ y^0 &= \frac{dS}{dx^0} = \frac{dS}{d\gamma}\frac{d\gamma}{dx^0} = \frac{1}{2}\sqrt{\frac{B}{A}}\frac{d\gamma}{dx^0}\int\limits_0^y \frac{dy}{\sqrt{\gamma + \cos y}}. \end{aligned} \tag{18}$$

The function γ is more or less constant because of the Jacobi integral F = const. /71

Below, we will assume that the value of the function γ never approaches a value of +1 too closely.

Two cases must be differentiated, depending on the value of γ.

Consider the first case in which

$$\gamma > +1.$$

Then, the argument y increases infinitely with y^0. [This happens if the square root in eqs.(18) is positive, which we assume here. If x were negative, it would be sufficient to select -x and -y as variables instead of x and y.] We will define the relation between γ and x^0 by expressing that y^0 and y increase simultaneously by 2π. From this, we obtain the condition

$$\begin{aligned} 2\pi &= \frac{1}{2}\sqrt{\frac{B}{A}}\frac{d\gamma}{dx^0}\int\limits_0^{2\pi}\frac{dy}{\sqrt{\gamma + \cos y}} \\ \gamma &= +1 \quad \text{for} \quad x^0 = 0. \end{aligned} \tag{19}$$

In view of this, x and y - y^0 are periodic in y^0, with the period 2π. In addition, the functions x and y depend in a certain manner of $\gamma(x^0)$ and are holomorphous as long as

$$x^0 > 0 \qquad \text{i.e.,}\ \gamma > +1.$$

Consider next the second case in which

$$-1 < \gamma < +1.$$

It is then necessary that

$$-\gamma \leq \cos y < +1.$$

Consequently, the argument y can never exceed the two values $\pm \cos^{-1}(-\gamma)$. In this case, we will define the function $\gamma(x^0)$ by the relation

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$$2x = \frac{1}{2}\sqrt{\frac{B}{A}\frac{d\gamma}{dx^0}} \int_0^{\arccos(-\gamma)} \frac{dy}{\sqrt{\gamma + \cos y}}, \tag{20}$$

$$\gamma = +1 \quad for \quad x^0 = 0.$$

Thus, the variables x and y, expressed as functions of x^0 and y^0, are periodic in y_0 with the period 2π; in addition, these functions remain holomorphous as long as

$$-c < x^0 < 0,$$

where c is the value of x^0 which corresponds to $\gamma = -1$.

In both these cases, the variables x^0, x'; y^0, y' satisfy a canonical system

$$\frac{dx^0}{d\tau} = \frac{d\Phi}{dy_0}, \qquad \frac{dy^0}{d\tau} = -\frac{d\Phi}{dx^0},$$
$$\frac{dx'}{d\tau} = \frac{d\Phi}{dy'}, \qquad \frac{dy'}{d\tau} = -\frac{d\Phi}{dx'}. \tag{21}$$

The characteristic function $\Phi(x^0, x'; y^0, y')$ is nothing else but the function $F(x, x'; y, y')$ after x and y have been replaced by their expressions as a function of $\gamma(x^0)$ and y^0, derived from the transformation (18). Thus, the function Φ is periodic with respect to y^0 and y', with the period 2π. In addition, Φ can be expanded in the form

$$\Phi = \Phi^{(0)} + \mu'\Phi^{(1)} + \mu'^2\Phi^{(2)} + \cdots. \tag{22}$$

The first term of this series has the expression

$$\Phi^{(0)} = B\gamma(x^0).$$

Consequently, the system (21) has the normal form given in Section 1. The method of reduction discussed there remains applicable here as long as the divisor

$$\frac{d\Phi^{(0)}}{dx^{(0)}} = B\frac{d\gamma}{dx^0} \tag{23}$$

does not become too small, i.e., as long as x_0 does not approach too closely /73 the value zero (or else as long as γ does not approach too closely the value +1).

To reduce the system (21), we start from the equation

$$\Phi\left(\frac{dS}{dy^0}, \frac{dS}{dy'}; y^0, y'\right) = \Phi_*\left(x^0, x'; \frac{dS}{dx'}\right).$$

The unknowns Φ_* and S can be expanded in powers of μ', in such a manner that

$$\begin{aligned} \Phi_* &= \Phi_*^{(0)} + \mu'\Phi_*^{(1)} + \mu'^2\Phi_*^{(2)} + \cdots, \\ S &= S^{(0)} + \mu' S^{(1)} + \mu'^2 S^{(2)} + \cdots. \end{aligned} \tag{24}$$

We set

$$S^{(0)} = x^0 y^0 + x' y'.$$

We find first

$$\Phi_*^{(0)} = \Phi^{(0)} = B\gamma(x^0).$$

Next, to determine $\Phi_*^{(1)}$ and $S^{(1)}$, we obtain the equation

$$\frac{d\Phi^{(0)}}{dx^0}\frac{dS^{(1)}}{dy^0} + \Phi^{(1)} = \Phi_*^{(1)}.$$

So that $S^{(1)}$ be periodic in y^0, it is necessary to put

$$\Phi_*^{(1)} = [\Phi^{(1)}] = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(1)}\, dy^0.$$

In view of this, the function $S^{(1)}$ is obtained after a quadrature and without small divisors, since we assumed that the derivative $\dfrac{d\Phi^{(0)}}{dx^0}$ is not small.

Under this condition, we can continue in the same manner and successively determine the various terms of the series (24).

Let us assume the following function as being formed: /74

$$S(x^0_*, x'_*; y^0, y').$$

By means of the canonical transformation

$$x^0 = \frac{dS}{dy^0}, \quad y^0_* = \frac{dS}{dx^0_*},$$
$$x' = \frac{dS}{dy'}, \quad y'_* = \frac{dS}{dx'_*}, \tag{25}$$

we introduce new variables x^0_*, x'_*; y^0_*, y'_* which satisfy the equations

$$\frac{dx^0_*}{d\tau} = 0, \qquad \frac{dy^0_*}{d\tau} = -\frac{d\Phi_*}{dx^0_*}, \tag{26}$$

$$\frac{dx'_*}{d\tau} = \frac{d\Phi_*}{dy'_*}, \quad \frac{dy'_*}{d\tau} = -\frac{d\Phi_*}{dx'_*}. \tag{27}$$

It is obvious that

$$x^0_* = \text{const.}$$

We assume that the value of $\gamma(x^0_*)$ is not too close to +1.

The variables x'_*, y'_* satisfy a canonical system (27) with one degree of freedom. Because of this, we obtain the relation

$$\Phi_*(x^0_*, x'_*; y'_*) = \bar{\varphi} = \text{const.} \tag{28}$$

Thus, x'_* is a certain function of y'_*. We must assume that the characteristic function Φ_* is such that the absolute value of x'_* always remains below a certain limit, comparable in magnitude to unity. In fact, for too large values of the quantity $|x'_*|$, the expansion of the function Φ_* in powers of the given parameter μ' could become illusory, since the various terms of this series are polynomials with respect to x'_*. Let $\Phi^{(k)}_*$ be the first of the functions $\Phi^{(1)}_*$, $\Phi^{(2)}_*$, ... which is not a constant. If the abbreviated relation /75

$$\Phi^{(k)}_*(x^0_*, x'_*; y'_*) = \text{const.} \tag{29}$$

defines a limited function x'_*, then the complete relation (28) will do the same. If, conversely, in the relation (29), the quantity x'_* may become infinite or else if $\Phi^{(k)}_*$ no longer depends on x'_*, then the integration method given here is no longer applicable.

Thus, let us assume that $|x'_*|$ always remains below a limit which is not too high. It is obvious that x'_*, $\cos y'_*$, and $\sin y'_*$ are periodic functions of τ. At the end of the period, the argument y'_* will either have increased by 2π or else this argument will have resumed the value it had at the beginning of the period.

The second of the relations in the system (26) indicates that the derivative of the argument y_*^0 is also periodic, with the same period.

Equations (25), solved first with respect to the unknowns x^0, x' ; y^0, y', will finally yield these variables as known functions of τ. We note specifically that the differences $x^0 - x_*^0$, $x' - x_*'$; $y^0 - y_*^0$, $y' - y_*'$ always remain small and of the order of μ'.

Finally, the transformation (18) will yield x and y as functions of τ.

Thus, in the hypotheses established by us as to the value of the parameter x_*^0 and as to the variations of x_*', the absolute values of the variables x and x' remain limited and comparable in magnitude to unity. Depending on the values selected for the integration constants x_*^0 and φ, one or the other of the two arguments y and y' can either increase (or decrease) infinitely with τ or else remain enclosed between two limits. In the latter case, the considered argument presents a libration.

Let us now return to the cases of the group $(\frac{1}{2})_3$. We must first find whether the function x_*' remains limited. In view of the second of the formulas (4) as well as of the transformation (13), it is obvious that the function $F^{(1)}$ has the particular form of

$$F^{(1)} = (ax + a'x')\cos y,$$

where a and a' are constants. The function $\Phi^{(1)}$ is obtained by introducing, in $F^{(1)}$, the quantities x and y as functions of x^0 and y^0 in accordance with the transformation (18). Finally, we find /76

$$\Phi_*^{(1)}(x^0, x'; y') = \frac{1}{2\pi}\int_0^{2\pi} F^{(1)}\, dy^0$$

$$= \frac{1}{2\pi}\frac{1}{2}\sqrt{\frac{B}{A}\frac{d\gamma}{dx^0}}\left\{a\sqrt{\frac{B}{A}}\int_{(c)}\cos y\, dy + a'x'\int_{(c)}\frac{\cos y\, dy}{\sqrt{\gamma + \cos y}}\right\}.$$

By means of integration (c), the variable y must increase from zero to 2π, if

$$\gamma(x^0) > +1.$$

Conversely, this variable must first increase from $-\cos^{-1}(-\gamma)$ up to $+\cos^{-1}(-\gamma)$ and then decrease by the same values, if

$$-1 < \gamma(x^0) < +1.$$

In the first case, the square root $\sqrt{\gamma + \cos y}$ remains positive while, in the second case, it will always have the same sign as dy. Consequently, the integral

$$\int_{(c)} \cos y \, dy$$

is zero in both cases. Finally, because of eqs.(19) and (20), we will have

$$\Phi_*^{(1)} = a' x' R(\gamma),$$

where

$$R(\gamma) = \int_{(c)} \frac{\cos y \, dy}{\sqrt{\gamma + \cos y}} : \int_{(c)} \frac{dy}{\sqrt{\gamma + \cos y}}.$$

The quantity $R(\gamma)$ is not identically zero.

Let us now find the expression for a'. The second of the formulas in the system (4) as well as the relations (13) show that /77

$$a' = \pm \frac{d\overline{h}_{1,j'}^{(1/2)}}{d\gamma}.$$

In Section 28, we gave the expression for the function $h_{1,j'}^{(1/2)}$ in the six cases of the group $(\frac{1}{2})_3$. We thus find that

$$a' = 0, \quad \text{if } \beta = 0,$$

and that

$$a' \neq 0, \quad \text{if } \beta = -3, -2, -1, +1, +2.$$

In the five cases in which $\beta \neq 0$, the variable x'_* is more or less constant. It follows from this that the discussed integration method is still applicable, provided that the value of γ is not too close to +1.

Conversely, in the remaining case of the group $(\frac{1}{2})_3$, at which $\beta = 0$, the function $\Phi_*^{(1)}$ vanishes identically. Evidently, the functions $F^{(1)}$, $\Phi^{(1)}$, and $S^{(1)}$ are also identically zero. It then becomes necessary to investigate the function $\Phi_*^{(2)}$.

On forming the equation, defining $\Phi_*^{(2)}$ and $S^{(2)}$, we see immediately that

$$\Phi_*^{(2)} = [\Phi^{(2)}].$$

Moreover, the third of the formulas in the system (4) as well as the relations (13), on putting there $j' = 0$, indicate that

$$\Phi^{(2)} = F^{(2)} = H^{(1)}_{**}.$$

To investigate the form of this function, for the case in question in which q = = 3, α = 1, β = 0, it is necessary to return to eq.(4) of Section 28. In the sum given there, the number j' must satisfy the conditions

$$j' = j'' + 2\beta = j'' = \text{even}, \quad -6 \leq j' + j'' \leq 0.$$

It follows from this that the number j' can assume only the two values j' = 0 and j' = -2. Consequently, we will have

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$$F^{(2)} = \overline{h^{(1)}_{0,0}} + h^{(1)}_{2,0} \cos 2y + h^{(1)}_{2,-2} \cos (2y - 2y').$$

To obtain $\Phi^{(2)}$, it is sufficient to replace, in $F^{(2)}$, the quantity y by its expression as a function of x^0 and y^0. Using then the mean value of $\Phi^{(2)}$ with respect to the argument y^0 (and writing x^0_*, y'_* instead of x^0 and y'), we find that $\Phi^{(2)}_*$ would have the form

$$\Phi^{(2)}_* = P(x^0_*) + Q(x^0_*) \cos 2y'_*,$$

where P and Q are certain functions of x^0_*. This is an expression that depends on y'_* but is independent of x'_*. Consequently, the variations in the function x'_* would be of the order of μ'^{-1}, so that the given integration method no longer is applicable to the case in which q = 3, α = 1, β = 0.

Let us now return to the cases of the groups $(1)_3$ and $(1)_4$, but excluding the two cases (9). The second equation in the system (6) as well as the relations (13) indicate that the function $F^{(1)}$ now has the particular form

$$F^{(1)} = cx + c'x' + (ax + a'x') \cos y + f(y, y').$$

The quantities c, c', a, a' are constants. The function $f(y, y')$ is independent of x and of x' and is periodic in y and y'. We have, specifically,

$$c' = \frac{d\overline{h^{(1)}_{0,0}}}{d\chi'}, \quad a' = \pm \frac{d\overline{h^{(1)}_{1,j'}}}{d\chi'}, \quad f(y, y') = \overline{H^{(1)}_{**}}.$$

The constants c' and a' are not identically zero. It is obvious, as in the discussion of the cases of the group $(\frac{1}{2})_3$, that the function $\Phi^{(1)}_*$ has the form

$$\Phi^{(1)}_* = Kx'_* + P(y'_*),$$

where K is a constant that generally is not zero, whereas the function $P(y'_*)$ is independent of x'_* and periodic in y'_*. Consequently, the value of x'_* generally remains limited. The mentioned integration method is applicable, provided that the value of γ is not too close to +1.

Let us finally return to the cases of the groups $(-\frac{3}{2})_3$ and $(-\frac{3}{2})_5$, but /79 excluding the cases in which $H_{**}^{(3/2)}$ contains several periodic terms. The second of eqs.(8) as well as the relations (13) then show that

$$F^{(1)} = cx + c'x',$$

where

$$c' = \frac{dh_{0,0}^{(1)}}{d\chi'}.$$

From this it follows that

$$\Phi_*^{(1)} = c'x'_* + \text{const.}$$

If c' is not zero, which we assume here, the value of x'_* remains more or less constant. The mentioned integration method is then applicable, provided that ν is not too close to +1.

We will now integrate eqs.(15) of Section 27 for the case of

$$q = 3,\ \alpha = 1,\ \beta = 0$$

which, until now, had been set aside.

It is convenient to put then

$$\chi = \mu^{1/4}\chi_1, \qquad \tau = \mu^{1/2}t_1 = \mu^{3/2}t,$$

$$H_{**} + \frac{\bar{\nu}'}{2}\chi_{**} - c = \mu^{1/2}G$$

and to retain χ' as variable, since its variations are comparable in magnitude to unity. As starting point, we thus obtain the equations

$$\mu^{1/4}\frac{d\chi_1}{d\tau} = \frac{dG}{dv}, \qquad \mu^{1/4}\frac{dv}{d\tau} = -\frac{dG}{d\chi_1},$$
$$\frac{d\chi'}{d\tau} = \frac{dG}{dv'}, \qquad \frac{dv'}{d\tau} = -\frac{dG}{d\chi'}. \tag{30}$$

In the expansion /80

$$G = G_0 + \mu^{1/4}G_{1/4} + \mu^{1/2}G_{1/2} + \cdots,$$

we will have

$$G_0 = A\chi_1^2 + 2H^{1/2,0,0}_{1,0,0}\cos v,$$

$$G_{1/4} = 0, \tag{31}$$

$$G_{1/2} = \Big/_{\chi=0} H^{(1)}_{**} = \tilde{h}^{(1)}_{0,0} + \tilde{h}^{(1)}_{2,0}\cos 2v + \tilde{h}^{(1)}_{2,-2}\cos(2v-2v').$$

The coefficient $H^{1/2,0,0}_{1,0,0}$ is a constant (see Section 28). The quantities $\tilde{h}^{(1)}_{j,j'}$ are derived from the functions $h^{(1)}_{j,j'}$ defined in Section 28, by putting there $\chi = 0$. These quantities are polynomials in χ'; $\tilde{h}^{(1)}_{0,0}$ is of the third degree, $\tilde{h}^{(1)}_{2,0}$ of zero degree, and $\tilde{h}^{(1)}_{2,-2}$ of the second degree.

For reasons of analogy, we introduce the following notations:

$$x = \chi_1, \qquad y = v, \qquad F = G,$$

$$x' = \chi', \qquad y' = v', \qquad \mu' = \mu^{1/4}. \tag{32}$$

This will yield the system

$$\mu'\frac{dx}{d\tau} = \frac{dF}{dy}, \qquad \mu'\frac{dy}{d\tau} = -\frac{dF}{dx},$$

$$\frac{dx'}{d\tau} = \frac{dF}{dy'}, \qquad \frac{dy'}{d\tau} = -\frac{dF}{dx'}. \tag{33}$$

The function F is given by the expansion

$$F = F^{(0)} + * + \mu'^2 F^{(2)} + \mu'^3 F^{(3)} + \cdots,$$

where

$$F^{(0)} \quad G_0 = Ax^2 - B\cos y$$

$$F^{(2)} = G_{1/2} = \tilde{h}^{(1)}_{0,0} + \tilde{h}^{(1)}_{2,0}\cos 2y + \tilde{h}^{(1)}_{2,-2}\cos(2y-2y'). \tag{34}$$

We next introduce new variables x^0, y^0 instead of x and y, by again making use of eqs.(18), (19), and (20). /81

Thus, the system (33) is replaced by

$$\mu'\frac{dx^0}{d\tau} = \frac{d\Phi}{dy^0}, \qquad \mu'\frac{dy^0}{d\tau} = -\frac{d\Phi}{dx^0}, \tag{35}$$

$$\frac{dx'}{d\tau} = \frac{d\Phi}{dy'}, \qquad \frac{dy'}{d\tau} = -\frac{d\Phi}{dx'}.$$

where the function

$$\Phi(x^0, x'; y^0, y')$$

represents what becomes the function

$$F(x, x'; y, y'),$$

if x and y are replaced by their expressions as functions of $\gamma(x^0)$ and y^0.

The function Φ is periodic in y^0 and y', with the period 2π. This function can be expanded in the form of

$$\Phi = \Phi^{(0)} + * + \mu'^2 \Phi^{(2)} + \mu'^3 \Phi^{(3)} + \cdots.$$

We have, specifically,

$$\Phi^{(0)} = B\gamma(x^0).$$

The function $\gamma(x^0)$ must also avoid the vicinity of the value +1.

To reduce the system (35), we start from the equation

$$\Phi\left(\frac{1}{\mu'}\frac{dS'}{dy^0}\ \frac{dS'}{dy'};\ y^0, y'\right) = \Phi_*\left(x^0, x';\ \frac{dS'}{dx'}\right),$$

where S' and Φ_* are unknown functions. We put

$$S' = x'y' + \mu'(x^0y^0 + S).$$

Then, the function S satisfies the equation /82

$$\Phi\left(x^0 + \frac{dS}{dy^0}, x' + \mu'\frac{dS}{dy'};\ y^0, y'\right) = \Phi_*\left(x^0, x';\ y' + \mu'\frac{dS}{dx'}\right).$$

It is possible to expand Φ_* and S in the form of

$$\begin{aligned} \Phi_* &= \Phi_*^{(0)} + * + \mu'^2 \Phi_*^{(2)} + \mu'^3 \Phi_*^{(3)} + \cdots, \\ S &= \qquad\qquad \mu'^2 S^{(2)} + \mu'^3 S^{(3)} + \cdots. \end{aligned} \tag{36}$$

We first find

$$\Phi_*^{(0)} = \Phi^{(0)} = B\gamma(x^0).$$

Then, the equation yielding $\Phi_*^{(2)}$ and $S^{(3)}$ becomes

$$\frac{d\Phi^{(0)}}{dx^0}\frac{dS^{(2)}}{dy^0} + \Phi^{(2)} = \Phi_*^{(2)}.$$

For $\Phi_*^{(2)}$, we must select the mean value of $\Phi^{(2)}$, i.e.,

$$\Phi_*^{(2)} = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{(2)} dy^0 = \tilde{h}_{0,0}^{(1)} + (\tilde{h}_{2,0}^{(1)} + \tilde{h}_{2,-2}^{(1)} \cos 2y')\, Q(\gamma), \tag{37}$$

where the function $Q(\gamma)$ is defined by the formula

$$Q(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \cos 2y\, dy^0 = \frac{1}{2\pi}\frac{1}{2}\sqrt{\frac{B}{A}}\frac{d\gamma}{dx^0}\int_{(c)} \frac{\cos 2y\, dy}{\sqrt{\gamma + \cos y}}$$

$$= \int_{(c)} \frac{\cos 2y\, dy}{\sqrt{\gamma + \cos y}} : \int_{(c)} \frac{dy}{\sqrt{\gamma + \cos y}}.$$

The integration procedure (c) has been defined on p.234. After having selected $\Phi_*^{(2)}$ in this manner, we obtain the function $S^{(2)}$ by a quadrature without small divisors, provided that the value of γ is not too close to 1.

Evidently, it is possible to continue in this manner and to successively /83 determine the various terms of the series (36).

Consider now the function

$$S'(x^0{}_*, x'_*;\ y^0, y')$$

as well as the canonical transformation

$$\mu' x^0 = \frac{dS'}{dy^0}, \qquad y^0{}_* = \frac{dS'}{\mu' dx^0{}_*},$$

$$x' = \frac{dS'}{dy'}, \qquad y'_* = \frac{dS'}{dx'_*},$$

which can also be written as

$$x^0 - x^0{}_* = \frac{dS}{dy^0}, \qquad y^0{}_* - y^0 = \frac{dS}{dx^0{}_*},$$

$$x' - x'_* = \mu' \frac{dS}{dy'}, \qquad y'_* - y' = \mu' \frac{dS}{dx'_*}. \tag{38}$$

In view of this transformation, we will have

$$\Phi(x^0, x';\ y^0, y') = \Phi_*(x^0{}_*, x'_*;\ y'_*).$$

Finally, the new variables satisfy the equations

$$\frac{dx'_*}{d\tau} = \frac{d\Phi_*}{dy'_*}, \qquad \frac{dy'_*}{d\tau} = -\frac{d\Phi_*}{dx'_*}, \tag{39}$$

$$\frac{dx^0_*}{d\tau} = 0, \qquad \frac{dy^0_*}{d\tau} = -\frac{d\Phi_*}{\mu' dx^0_*}. \tag{40}$$

Thus, x^0_* is a constant. Its value must be selected such that $\gamma(x^0_*)$ is not close to +1.

The variables x'_* and y'_* satisfy the canonical system (39) with one degree of freedom. This system has the first integral

$$\Phi_* = \Phi_*^{(0)} + \mu'^2 \Phi_*^{(2)} + \cdots = \varphi = \text{const.} \tag{41}$$

The first term $\Phi_*^{(0)}$ depends only on x^0_* and is a constant. In accordance with /84 eq.(37), the function $\Phi_*^{(2)}$ has the form

$$P(x'_*) + P_1(x'_*) \cos 2y'_*,$$

where P is a polynomial of the third degree and P_1 is a polynomial of the second degree in x'_*.

The relation (41) shows that the variable x'_* always remains limited and comparable in magnitude to unity.

In addition, the quantities x'_*, $\cos y'_*$, and $\sin y'_*$ are periodic functions of τ. This is the same for the derivative of the argument y^0_*, in accordance with the second equation of the system (40).

After this, let us return to the relations (38). These equations can be solved for x^0, x' ; y^0, y' by making use of the generalized Lagrange method with two variables y^0 and y'. We conclude, specifically, that the differences $x^0 - x^0_*$ and $x' - x'_*$ are comparable to $\mu'^2 = \mu^{1/2}$ resp. $\mu'^3 = \mu^{3/4}$. Thus, x^0 is more or less constant, and the value of x' remains limited.

Finally eqs.(18) and (19) or eq.(20) permit expressing x and y as functions of $\gamma(x^0)$ and y^0.

Thus, the system (33) or the system (30) or else the system (15) of Section 27 can be integrated in the case in which $q = 3$, $\alpha = 1$, $\beta = 0$, provided that γ is not too close to +1.

In the method used for integrating the systems (14) and (33), we assumed that the value of the quantity γ, introduced by eqs.(18) and (19) or eq.(20), is not too close to +1. We found that the series (24) and (36) actually contain negative powers of $\gamma - 1$ and of $\log|\gamma - 1|$ such that their first terms converge more or less like the terms of a geometric series, in accordance with powers of the ratio $\mu' : (\gamma - 1)$. Consequently, if $|\gamma - 1|$ is of the order of $\mu^{1/4}$ in the investigated cases of the groups $(\frac{1}{2})_3$, $(\frac{3}{2})_3$, and $(\frac{3}{2})_5$ and also if $|\gamma - 1|$ is comparable to $\mu^{1/2}$ in the investigated cases of the groups $(1)_3$ and $(1)_4$, it is impossible to say anything as to the form of the solution of the system (15) in Section 27. Specifically, we do not know then whether the variable χ' still remains limited and whether the unknown χ constantly remains small.

We should mention also the variations of the arguments in the cases investigated in this Section 30. We have seen that one or the other of the arguments /85

$$v + j'v' \quad \text{and} \quad v'$$

(or both simultaneously) sometimes is subject to a libration. To these librations there obviously correspond the relations

$$\alpha\{pn - (p + q)n'\} + j'\mu\nu' + (j' - \beta)\mu\nu'' = 0,$$

$$\mu\nu' + \mu\nu'' = 0$$

between the mean motions n, n', $-\mu\nu'$, and $-\mu\nu''$ defined above (see pp.196 and 227). Here, we encounter, for the first time in our report, cases characterized by two simultaneous librations.

A few words should be said on the excluded cases of the groups $(1)_q$ resp. $(\frac{3}{2})_q$ in which the function $H^{(1)}_{**}$ resp. $H^{(3/2)}_{**}$ contain several periodic terms. Let us consider, for example, the case

$$q = 4, \ \alpha = 1, \ \beta = 0. \tag{42}$$

If we also use eqs.(1) of Section 30, we will obtain a system of the form (2) in which

$$G_0 = A\chi_1'^2 - B\cos v - B'\cos(v - 2v').$$

In this expression, A, B, and B' are constants.

The canonical system for $\mu = 0$, whose characteristic function is G_0, can be integrated by the Jacobi method. However, it is easy to demonstrate that, in this solution, the variable χ_1' is not limited. Evidently, we cannot start from such a solution, in the application of the method of arbitrary variation of constants.

The artifice used so successfully in the case in which $q = 3$, $\alpha = 1$, $\beta = 1$ is no longer efficient in the present case.

Consequently, in the excluded cases in which G_0 contains several periodic terms, we know nothing as yet on the general character of the solution, if the initial value of χ is sufficiently small. We will return to this question in Section 32 and treat it there in a different manner. /86

Section 31.

Toward the end of Section 29, we integrated the system (15) of Section 27, by assuming that the initial value of χ is of the order of μ or smaller. We excluded there not only the cases investigated later in the present Section but also the cases of the groups $(2)_q$, i.e., the cases in which the argument v appears first in the function $H^{(2)}_{**}$. Here, we will resume the integration for certain cases of the groups $(2)_q$. We again will use eqs.(8) of Section 29. The variables χ_1, χ_1'; v, v' again satisfy a canonical system of the form (9) of Section 29. In the series (10) of the same Section, the first term will now be expressed by

$$G_0 = A\chi_1^2 + \frac{d\bar{h}^{(1)}_{0,0}}{d\chi}\chi_1 + \frac{d\bar{h}^{(1)}_{0,0}}{d\chi'}\chi_1' + \bar{h}^{(2)}_{1,j'}\cos(v + j'v').$$

Here, we have already caused the following term to vanish by using the artifice given on p.226

$$\bar{h}^{(2)}_{0,2}\cos 2v'$$

which does appear as soon as q = 3 or 4. In addition, we have assumed that the function $H^{(2)}_{**}$ includes a single term periodic in v.

We will introduce here new variables by setting, somewhat similar to eq.(13) of Section 30,

$$\begin{aligned} x &= \chi_1 + \frac{1}{2A}\left(\frac{d\bar{h}^{(1)}_{0,0}}{d\chi} + j'\frac{d\bar{h}^{(1)}_{0,0}}{d\chi'}\right), & y &= v + j'v', \\ x' &= -j'\chi_1 + \chi_1', & y' &= v'. \end{aligned} \tag{1}$$

The variables x, x'; y, y' satisfy the canonical system /87

$$\begin{aligned} \frac{dx}{dt} &= \frac{dF}{dy}, & \frac{dy}{d\tau} &= -\frac{dF}{dx}, \\ \frac{dx'}{dt} &= \frac{dF}{dy'}, & \frac{dy'}{d\tau} &= -\frac{dF}{dx'}. \end{aligned} \tag{2}$$

The characteristic function F = G - const can be expanded in powers of $\mu' = \mu^{1/2}$:

$$F = F^{(0)} + \mu' F^{(1)} + \mu'^2 F^{(2)} + \cdots. \tag{3}$$

The principal term of this expansion has the specific form

$$F^{(0)} = Ax^2 - B\cos y + Cx'. \qquad (4)$$

The quantities A, B, C are constants. The last quantity is expressed by

$$C = \frac{dh^{(1)}_{0,0}}{d\chi'}. \qquad (5)$$

The functions $F^{(1)}$, $F^{(2)}$, ... are polynomials in x and x', which are periodic with respect to the angular variables y and y' having the period 2π.

This makes us return again to a canonical system, slightly more general than the system (14) in Section 30. Below, we will assume that

$$C \neq 0,$$

and even that $|C|$ is not small.

To reduce eqs.(2) to the normal form given in Section 1, we will replace x and y by the variables x^0 and y^0, by means of the canonical transformation (18) of Section 30.

The function $\gamma(x^0)$ is practically constant because of the Jacobi integral and because of the fact that x' is practically constant. Below, we will assume that the value of γ does not too closely approach the value +1.

As in Section 30, we differentiate two cases. If /88

$$\gamma > +1,$$

then the function γ will be defined by eq.(19) of Section 30. If, conversely,

$$-1 < \gamma < +1,$$

then γ will be determined by eq.(20) of the same Section.

In the first case, x and $y - y^0$ are periodic functions in y^0, with the period 2π. In the second case, x and y resume their initial values as soon as y^0 increases by 2π. In addition, x and y are holomorphous in x^0 as long as $\gamma(x^0) \neq +1$.

The variables x^0, x' ; y^0, y' satisfy a canonical system

$$\frac{dx^0}{d\tau} = \frac{d\Phi}{dy^0}, \qquad \frac{dy^0}{d\tau} = -\frac{d\Phi}{dx^0},$$
$$\frac{dx'}{d\tau} = \frac{d\Phi}{dy'}, \qquad \frac{dy'}{d\tau} = -\frac{d\Phi}{dx'}, \qquad (6)$$

where

$$F(x, x'; y, y') = \Phi(x^0, x'; y^0, y').$$

The function Φ, which is periodic in y^0 and y', with the period 2π, can be expanded in the form of

$$\Phi = \Phi^{(0)} + \mu'\Phi^{(1)} + \mu'^2\Phi^{(2)} + \cdots.$$

The principal term of this expansion is expressed by

$$\Phi^{(0)} = B\gamma(x^0) + Cx'.$$

Thus, the system (6) is of the normal type, investigated in Section 1.

If the two quantities

$$B\frac{d\gamma}{dx^0} \quad \text{and} \quad C$$

are not more or less at a commensurable and simple ratio, then the system (6) can be reduced to another system whose characteristic function does not depend on the angular variables. In that case, a formal integration is easy. /89

If, conversely, a certain divisor

$$kB\frac{d\gamma}{dx^0} + k'C$$

is small, we can use the method described in Section 1 for introducing new canonical variables x^0_*, x'_*; y^0_*, y'_*. The new characteristic function depends on y^0_* and y'_* only in the combination

$$ky^0_* + k'y'_*.$$

From this, we obtain the first integral

$$k'x^0_* - kx'_* = \text{const.}$$

By means of a linear canonical transformation, the new system reduces to another system, with one degree of freedom. It is easy to demonstrate that the formal integration presents no difficulty.

Thus, we are able to integrate eqs.(15) of Section 27 also in the cases of the groups $(2)_q$, by assuming that the initial value of χ is of the order of μ or smaller, that the initial value of χ' is neither too large nor too small, and that - in addition - the initial value of the derivative

$$\frac{dh^{(1)}_{0,0}}{d\chi'}$$

is not too small. (We have been forced to exclude only the cases in which the function $H^{(2)}_{**}$ includes two or several periodic terms in v.) Under these conditions, we found that the ratio $|\chi|:\mu$ cannot become large and that the value of χ' remains more or less constant.

Section 32.

In the three preceding Sections, we have frequently been able to complete the integration of the equations of motion of doubly critical planets. In all cases in which the integration has been possible, it happens that the major axis, the eccentricity, and the inclination remain practically invariant. /90

This raises an interesting problem. It is here a question to know whether, in the cases in which formal integration has been impossible, the inequalities of the mentioned elements may become so extensive that the investigated planet changes into a comet.

To investigate this problem, we return to eqs.(15) of Section 27. The canonical system in question possesses the first Jacobi integral

$$H_{**} = \text{const.} \tag{1}$$

Primarily, let us consider the types in which

$$q = 3$$

excluding, however, the cases of the group $(1/2)_3$, i.e., the cases

$$q = 3, \quad \alpha = 1, \quad -3 \leq \beta < +2. \tag{2}$$

Let

$$(H^{(1)}_{**})$$

be the expression of the function $H^{(1)}_{**}$ by putting there $\chi = 0$.

The Jacobi integral can then be written as

$$A\chi^2 + \mu(H^{(1)}_{**}) + \cdots = \text{const.} \tag{3}$$

The neglected terms of the first member of this equation are of the order of $\mu^{3/2}$ or smaller, as long as $|\chi|:\sqrt{\mu}$ and χ' have values that are not too large. The function $(H^{(1)}_{**})$ is a polynomial in $\sqrt{\chi'}$, with coefficients that are trigonometric functions with respect to the arguments v and v' in the cases of the group $(1)_3$ and are constants in the cases of the groups $(\frac{3}{2})_3$, $(2)_3$, $(\frac{5}{2})_3$, $(3)_3$ ... .

Let us recall specifically that the function $(H^{(1)}_{**})$, for $q = 3$, is a polynomial of the sixth degree in $\sqrt{\chi'}$ and that the coefficient of χ'^3 is always positive

(see p. 213).

In accordance with our assumption, the constant of the second member of eq.(2) is of the order of μ. In view of this, it is obvious that the values of the positive quantities $|\chi|:\sqrt{\mu}$ and χ' can never exceed a certain finite limit /91
which is independent of μ.

Thus, for critical planets of the types $\frac{p+3}{p}$, the stability is ensured from the formal viewpoint, except perhaps in the cases (2).

This proof is no longer applicable at $q \geq 4$, since then the function $H^{(1)}_{**}$ may become negative for large values of χ'.

Let us now consider the types in which

$$q \geq 4$$

excluding only the cases of the group $(1)_4$, i.e., the cases

$$q = 4, \quad \alpha = 1, \quad \beta = -4, -3, -2, -1, 0, +1, +2, +4. \tag{4}$$

We will assume here that, at the origin of time, the absolute value of the unknown χ is small with respect to $\mu^{1/2}$ and that the value of χ' is finite. We will demonstrate that the absolute value of χ will always remain small with respect to $\mu^{1/2}$ and that the unknown χ' will always remain practically invariant.

For the proof, we again start from the first Jacobi integral, given by eq.(1).

In the series (17) of Section 27, the first terms are expressed by

$$H^{(0)}_{**} = A\chi^2 + \text{const.},$$
$$H^{(1/2)}_{**} = 0,$$
$$H^{(1)}_{**} = h^{(1)}_{0,0}.$$

With respect to the function $h^{(1)}_{0,0}$, it is sufficient to recall that this function is a polynomial in χ', which is independent of v and v', is rational with respect to χ, and is finite for $\chi = 0$.

Let $(h^{(1)}_{0,0})$ be the polynomial in χ' with constant coefficients, obtained by putting $\chi = 0$ in $h^{(1)}_{0,0}$.

Then, because of the Jacobi integral, we will have /92

$$A\chi^2 + \mu(h^{(1)}_{0,0}) + \cdots = \text{const.}$$

The neglected terms of the first member of this equation are of the order of

$\mu^{3/2}$, as long as χ' and $|\chi : \sqrt{\mu}|$ have not too large a value.

Let us assume, an assumption which we want to prove impossible, that the variable χ' varies by a quantity comparable in magnitude to unity. In a case of this nature, we could fix an instant t_0' such that the value of χ' would be neither too large nor too small but sufficiently different from the initial value. Because of the Jacobi integral, the corresponding value of the ratio $|\chi : \sqrt{\mu}|$ would be neither too large nor too small but comparable in magnitude to unity. Then, we could select the instant t_0' as the origin of time and demonstrate, by means of the process given in Section 29, that the absolute value of the unknown χ could still have been comparable in magnitude to unity. However, this cannot be true since we have assumed that χ had first been small with respect to $\sqrt{\mu}$. Thus, χ' remains more or less invariant, and the quantity $|\chi| : \sqrt{\mu}$ remains still small.

From the above discussion we can conclude that the inequalities of the major axis, of the eccentricity, and of the inclination are always small for doubly critical planets, except possibly for planets of the types (p + 3):p in the cases in which

$$\alpha = 1, \quad \beta = -3, -2, -1, 0, +1, +2,$$

and for planets of the types (p + 4):p in the cases in which

$$\alpha = 1, \quad \beta = -4, -3, -2, -1, 0, +1, +2, +4.$$

In these exceptional cases, we are unable to make any statements as to the magnitude of the inequalities of the mentioned elements.

We want to emphasize specifically the rather remarkable result that the motion of critical planets is stable from the viewpoint of formal calculus, in all types in which

$$q \geq 5.$$

Before concluding this report, we will give a few formulas that might be useful for classifying, in the theory of doubly critical planets, the inequalities of the primary elements x_1, y_1, ξ_k, η_k defined in Section 2. These formulas are /93

$$\begin{aligned} x_1 &= \dot{x}_1 + (x_1 - \dot{x}_1) = \dot{x}_1 + \mu z + (x_1 - \dot{x}_1) \\ &= \dot{x}_1 + \mu z_{**} + \mu(z - z_{**}) + (x_1 - \dot{x}_1) \\ &= \dot{x}_1 + \mu \bar{z}_{\alpha,\beta} + \mu \alpha p \chi + \mu(z - z_{**}) + x_1 - \dot{x}_1, \end{aligned}$$

$$\begin{aligned} y_1 &= \dot{y}_1 + (y_1 - \dot{y}_1) = \dot{y} + \frac{p+q}{p} t + (y_1 - \dot{y}_1) \\ &= \dot{y}_{**} + \frac{p+q}{p} t + (\dot{y} - \dot{y}_{**}) + (y_1 - \dot{y}_1), \end{aligned}$$

$$\xi_1 = \dot{\xi}_1 + (\xi_1 - \dot{\xi}_1) = \bar{c} e' + \sqrt{\mu}\, \xi'_{**} + \sqrt{\mu}(\xi' - \xi'_{**}) + (\xi_1 - \dot{\xi}_1),$$

$$\eta_1 = \dot{\eta}_1 + (\eta_1 - \dot{\eta}_1) = \sqrt{\mu}\,\eta'_{**} + \sqrt{\mu}\,(\eta' - \eta'_{**}) + (\eta_1 - \dot{\eta}_1),$$

$$\xi_2 = \dot{\xi}_2 + (\xi_2 - \dot{\xi}_2) = \sqrt{\mu}\,\xi''_{**} + \sqrt{\mu}\,(\xi'' - \xi''_{**}) + (\xi_2 - \dot{\xi}_2),$$

$$\eta_2 = \dot{\eta}_2 + (\eta_2 - \dot{\eta}_2) = \sqrt{\mu}\,\eta''_{**} + \sqrt{\mu}\,(\eta'' - \eta''_{**}) + (\eta_2 - \dot{\eta}_2).$$

Thus, the inequalities are subdivided into three groups. Those of the first group are slowly variable and are obtained by integration of the system (11) of Section 27. The inequalities of the second group, which also vary slowly, are derived after solving eqs.(10) of the same section. Finally, the inequalities of the third group, whose variations are rapid, are defined by the formulas (13) of Section 16.

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